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Non-Markov Gaussian Term Structure Models: The Case of Inflation*

BRUNO FEUNOU and JEAN-SÉBASTIEN FONTAINE
Bank of Canada

Abstract. Standard Gaussian term structure models impose the Markov property: the conditional mean is a function of the risk factors. We relax this assumption and consider models where yields are linear in the conditional mean (but not in the risk factors). To illustrate, yields should span expected inflation but not inflation. Second, expected and surprise yield changes can have opposite contemporaneous effects on expected inflation. Third, the survey forecasts and inflation rate can both be in the state. These three features are inconsistent with the Markov assumption. These effects matter empirically in the USA and in Canada.

JEL Classification: E43, E47, G12

1. Introduction

Standard affine macro-finance term structure models (MTSMs) require that yields be linear functions of the risk factors $\mathcal{Z}_t \in \mathbf{R}^{\mathcal{N}_Z}$—the spanning property—and that the conditional mean of the future risk factors $\mathcal{E}_t \equiv E_t[\mathcal{Z}_{t+1}]$ be a linear function of the same risk factors—the Markov property.[1] See, for instance, Joslin, Singleton, and Zhu (2011) and Joslin, Le, and Singleton (2013a), JSZ and JLS, hereafter. We introduce the family of Conditional Mean MTSM (CM-MTSMs) where (i) yields span $\mathcal{E}_t$ but not $\mathcal{Z}_t$ and where (ii) $\mathcal{Z}_t$ does not have the Markov property but $\mathcal{E}_t$ does.

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[1] The Markov property requires that the conditional distribution of $\mathcal{Z}_{t+1}$ be a function $\mathcal{Z}_t$, but this is equivalent to requiring that the conditional mean be a function of $\mathcal{Z}_t$ in homoscedastic Gaussian MTSMs.

The standard model leaves no separate role for the conditional mean in the behavior of yields—$\mathcal{E}_t$ is a linear function of $\mathcal{Z}_t$. We loosen this link parsimoniously, offering three key contributions.

First, the model implies that yields span expectations about future $\mathcal{Z}_t$ but not the current value. The assumption that yields span the current value $\mathcal{Z}_t$ is hardly supported in the data. For instance, Joslin, Priebsch, and Singleton (2014) (JPS) argue that current macro variables are weakly related to current yields and impose restrictions on the pricing kernel to limit the information span of yields.[2] Similarly, Duffee (2011) suggests that factors with opposite effects on expectations and on the risk premium can be hard to measure. We provide an alternative resolution where, in the example of inflation, bond yields should span inflation expectations but not inflation itself. This distinction is blurred in Markovian models.

Second, $\mathcal{E}_t$ mixes information from the innovations $u_t \equiv \mathcal{Z}_t - \mathcal{E}_{t-1}$ and the information contained in prior expectations $\mathcal{E}_{t-1}$, allowing in a direct and natural way for the opposite effects of expectations and innovations on future macro variables (Phelps, 1967; Friedman, 1968). For instance, expected and unexpected yield changes, including unexpected policy actions, affect future inflation with opposite signs. The weights on u_t and $\mathcal{E}_{t-1}$ are equal if (and only if) $\mathcal{Z}_t$ has the Markov property, nesting standard MTSMs. In addition, allowing for different weights implies that $\mathcal{E}_t$ is ultimately a function of the entire history of $\mathcal{Z}_t$—much as the conditional variance is a function of past returns in a GARCH model. Therefore, CM-MTSMs parsimoniously capture the fact that the inflation dynamics involve several lags of inflation (Kim, 2007) and of other macro variables (Ang, Piazzesi, and Wei, 2006).

Third, CM-MTSMs reconcile two strands of the literature. The first uses observed macro variables as risk factors driving yields (inflation, say). The second uses (inflation) survey forecasts (Piazzesi and Schneider, 2009; Chun, 2011). However, in the latter case, shocks to macro variables are left outside the model and the corresponding risk premium (the inflation-risk premium) is unidentified. The CM model maintains the connection between inflation and its conditional mean. We can easily include both inflation and survey inflation forecasts within the non-Markovian state dynamics and guarantee that the survey forecast corresponds to the model-implied conditional mean, up to some measurement error. In a Markovian model, this is only possible if

[2] In their empirical implementation, JPS use the Chicago Fed National Activity Index to measure real economic activity, and surveys of professional forecasters by Blue Chip Economic Indicators to measure inflation. In effect, JPS assume that yields do not span expected inflation.

measurement errors are zero, whereas the survey forecast is the model conditional mean.[3]

We use predictive regressions to argue that yields nearly span inflation expectations, to motivate that yields span $\mathcal{E}_t$ up to a measurement error. Since $\mathcal{E}_t$ has the Markov property, it suffices to know $\mathcal{E}_t$ to forecast any future $\mathcal{Z}_{t+h}$ using the law of iterated expectations. This requires model forecasts to be efficient but not necessarily the same as the survey forecasts. We check below that model forecasts match the accuracy of survey forecasts out-of-sample. Empirically, we estimate models with two yield factors and two macro variables—the first two principal components (PCs) from yields, the inflation rate and the unemployment rate. We estimate the CM model for the USA, using nominal yields, survey forecasts, and inflation swaps. We estimate the same models for Canada, using nominal yields and survey data, as well as a constraint on the inflation Sharpe ratio, to pin down the inflation–risk premium. For both countries, we also estimate the restricted Markovian VAR. The results make clear why relaxing the Markovian assumption is important to study the interaction between the macroeconomy and bond yields.

First, the estimates imply that yields span macro expectations in the CM model. Loadings on expected inflation range from 1.16 to 0.61 between the short rate and the 10-year bond, and loadings on expected unemployment range between -0.36 and -0.20. In contrast, these loadings are essentially zero in the VAR. In addition, the CM model produces the best inflation forecasts, supporting the assumptions that yields span $\mathcal{E}_t$. Forecasts from the CM model and forecasts from surveys produce similar out-of-sample root mean squared errors (RMSEs) for both Canada and the USA. Relative to the VAR, the CM model reduces the RMSE by as much as 20% when forecasting 2-year cumulative inflation.

Second, inflation expectation updates use very different weights on the prior expectations $\mathcal{E}_{t-1}$ and on the innovations u_t. The weight on lagged expected inflation is close to one, but the weight on inflation innovation is close to zero. Similarly, expected and surprise yield changes affect inflation with opposite signs. Expected inflation increases with an expected steepening of the curve but decreases with a surprise tightening of the curve. Expected inflation decreases with higher expected unemployment but increases with unemployment shocks. The VAR model cannot differentiate between these

[3] Other cases exist, but these restrict the rank of the state covariance matrix or the rank of its auto-regressive matrix: either one state or one of the conditional means is not independent.

effects; the loading estimates are close to zero, wrongly implying that yields do not span expected inflation.

Survey data help reduce sampling uncertainty and lessen the bias in persistence parameters.[4] In addition, survey forecasts are difficult to improve and using survey data helps forecast inflation (Ang, Bekaert, and Wei, 2007; Faust and Wright, 2011). We use inflation swaps to pin down the inflation–risk premium.[5] We study the robustness of the CM and VAR results obtained when using yields, inflation swaps, and inflation survey forecasts for estimation (Case A) to cases where we exclude inflation swaps (Case B) and where we also exclude survey forecasts (Case C). When excluding swap data (Cases B and C), or when these data are unavailable, as in Canada, we constrain the Sharpe ratio to plausible values, following Duffee (2011).[6] The CM model generates similar yield decompositions across cases. In contrast, the VAR estimates of expected inflation change dramatically when we exclude survey data, as do the estimates of the inflation-risk premium when we exclude inflation swaps. Therefore, the CM model can be used when the real and nominal markets are not integrated, whereas real bonds incorporate a liquidity premium (e.g., Campbell, Shiller, and Viceira, 2009; Fleckenstein, Longstaff, and Lustig, 2013), or when survey data are not available at the sampling frequency.

We provide the first risk-adjusted decomposition of nominal yields in Canada. The average nominal yield curve is upward-sloped, but the average real curve is downward-sloped. We find that the slope of nominal yields is due to the upward-sloped inflation-risk premia. In addition, real yields are more cyclical than nominal yields in Canada because the inflation-risk premium is countercyclical. In other words, inflation shocks are perceived as risky at the bottom of the cycle. This amplifies the transmission of policy rate decisions to the real curve. This contrasts with results for the USA where the inflation-risk premium is procyclical, dampening the effect of policy rate changes (Chernov and Mueller, 2011). One conjecture ties this difference to the different mandates given to each central bank.

[4] Kim and Wright (2005); Kim and Orphanides (2012); Bauer, Rudebusch, and Wu (2012); Jardet, Monfort, and Pegoraro (2013).

[5] Recent papers also use additional instruments: Chen, Liu, and Cheng (2010); Christensen, Lopez, and Rudebusch (2010); D'amico, Kim, and Wei (2010); Chernov and Mueller (2011); Haubrich, Pennacchi, and Ritchken (2012). Ajello, Benzoni, and Chyruk (2012) combine core, food, and energy inflation.

[6] See also Chernov and Mueller (2011) who limit the variability of the pricing kernel, and Bauer and Diez de los Rios (2012), who limit the Sharpe ratio in the context of a multicountry term structure model. A very tight constraint on the inflation Sharpe ratio (as in Ang, Bekaert, and Wei, 2008) is rejected in the data.

JSZ, JLS, and Chun (2011) are the most closely related papers. In our case, the state $\mathcal{Z}_t$ spans the macro variables $\mathcal{M}_t$, while $\mathcal{E}_t$ spans the cross-section of yields. In JLS, the state $\mathcal{Z}_t$ spans macro variables and yields. Nonetheless, the family of CM-MTSMs nests the JLS model up to a matrix Σ^* controlling the weights put on u_t and $\mathcal{E}_{t-1}$ in updating expectations. The CM-MTSM family also nests JSZ's yield-only dynamic term structure models (DTSMs) if no macro variables are included in the model. If we use survey forecasts as observations for $\mathcal{E}_t$, excluding other macro variables, we have the framework of Chun (2011).

The decomposition of nominal yields has a long history going back to Fisher (1930). Recent work includes Pennacchi (1991), Ang, Bekaert, and Wei (2008), and the references therein. Kim (2007) provides insight into some of the challenges involved. Monfort and Pegoraro (2007) also consider non-Markovian term structure models, but with regimes and with finite lags. They find that two lags of the yield factors match the predictability evidence. Joslin, Le, and Singleton (2013b) consider an asymmetric formulation where the yield factors have the Markov property under the pricing measure but not under the historical measure. Feunou and Meddahi (2007) analyze more general departures from the Markov assumption.

The remainder of this article is organized as follows. Section 2 follows and introduces the model and discusses its properties in relation to existing specifications. Section 3 describes the data and the estimation method. Section 4 reports the results. Section 5 concludes. All proofs are provided in the Appendix.

2. A Macro-Finance Conditional Mean Model

2.1 SPANNING SURVEY DATA WITH YIELDS

Our approach has that conditional means drive bond prices. Conversely, bond yields span conditional means, up to measurement errors. Before we proceed with the formal model development sections, we first provide a simple and direct assessment of whether yields span the conditional mean of inflation, which is the focus of our empirical implementation. We use survey forecasts to proxy for conditional expectations in a contemporaneous regression:

$$\mathcal{E}_{t,h}^{\mathrm{SPF}} = \alpha_h + \beta_h' PC_{n,t} + e_{t,h}, \tag{1}$$

where $PC_{n,t}$ is a vector stacking the first n PCs of yields and where $\mathcal{E}_{t,h}^{\mathrm{SPF}}$ is the median forecast from the Blue Chip Financial Forecast (BC) Survey observed at time t for the horizon h. We use up to four PCs from yields,

Table I. Spanning regressions

Panels A and B: R^2s from contemporaneous regression of survey forecasts on PCs from yields. Survey forecasts cover the current quarter up to four quarters ahead, Q0, Q1, Q2, Q3, and Q4, respectively. We use vectors of PCs that include either one, two, three, or four PCs from yields, $PC_{1,t}$, $PC_{2,t}$, $PC_{3,t}$, and $PC_{4,t}$. Panels C and D: R^2s from similar regressions for quarterly all-item urban CPI inflation rate and quarterly unemployment rate. Monthly survey data from Blue Chip, Gurkaynak, Sack, and Wright yield data, 1985–2012.

Panel A: spanning CPI inflation forecasts					Panel B: spanning unemployment forecasts				
	$PC_{t,1}$	$PC_{t,2}$	$PC_{t,3}$	$PC_{t,4}$		$PC_{t,1}$	$PC_{t,2}$	$PC_{t,3}$	$PC_{t,4}$
Q0	31.5	31.7	31.9	32.2	Q0	7.6	44.7	49.9	56.3
Q1	71.5	71.5	71.9	73.3	Q1	8.4	44.5	50.9	57.5
Q2	80.6	81.1	82.3	83.6	Q2	8.5	42.5	50.3	57.7
Q3	82.0	83.6	84.8	86.4	Q3	8.2	39.9	48.5	56.7
Q4	82.3	85.3	86.9	88.2	Q4	7.5	37.5	46.3	55.7
Panel C: spanning CPI inflation rate					Panel D: spanning unemployment rate				
	$PC_{t,1}$	$PC_{t,2}$	$PC_{t,3}$	$PC_{t,4}$		$PC_{t,1}$	$PC_{t,2}$	$PC_{t,3}$	$PC_{t,4}$
Obs.	10.8	11.4	13.4	13.4	Obs.	7.8	44.5	50.6	57.4

and consider forecasts of quarterly CPI inflation and the unemployment rate where the horizon ranges between the current quarter and up to four quarters ahead.[7] We use the regression's R^2 to measure the spanning relationship.

Panel A of Table I reports the results in the case of inflation. The R^2s from the inflation forecast range between 72% and 82% when using the first component of yields, $PC_{1,t}$. We find similar results using the quarterly Survey of Professional Forecasters (unreported). Adding other PCs widens that range to 73–88%. Forecasts for the current quarter provide an exception to the rule, with an R^2 of 32%. Forecasts of the current quarter capture realized changes in CPI, which are volatile and often idiosyncratic. It seems unlikely that yields are affected by these variations. For instance, Panel C shows that yields span only up to 13% of quarterly CPI inflation.

Although unemployment is not our focus, Panel B of Table I reports the results for unemployment forecasts. The R^2s reach 40% when two of the yield components are included, and up to 57% when all four components

[7] We match the BC forecasts with yield components observed on the 15th of the middle month of each quarter (or the preceding business day). The exact survey date has changed over time.

are included. Panel D shows that yields also span 57% of the realized unemployment rate.

The fact that yields span as much as 88% of inflation forecasts and close to 60% of unemployment forecasts, but as little as 13% of observed inflation, motivates a term structure model where yields span conditional means, which is the focus of this article.[8] Note that we only require that yields provide a valid conditional expectation. The above R^2 is not 1 and the data may not support the restriction that the expectations implicit in yields are equal to the survey forecasts. Indeed, survey participants do not report identical forecasts, exhibiting a fair degree of disagreement (see, e.g., Mankiw, Reis, and Wolfers, 2003), and there is no reason that the median response should agree with the investors' expectations implicit in yields. Hence, we require in the following that the model and survey forecast errors differ by an unpredictable component. We also check that model and survey forecasts have similar accuracy.

2.2 CONDITIONAL EXPECTATIONS AS YIELD FACTORS

We build the family of CM-MTSM starting with the conditional distribution of the latent factor $\mathcal{Z}_t \in \mathbf{R}^{\mathcal{N}_Z}$. Its conditional mean under the historical measure $\mathbb{P}$,

$$\mathcal{E}_t \equiv E_t[\mathcal{Z}_{t+1}], \tag{2}$$

drives the one-period interest rate i_t:

$$i_t = \rho_0 + \rho_1' \mathcal{E}_t. \tag{3}$$

The dynamics of $\mathcal{E}_t$ under the risk-neutral measure $\mathbb{Q}$ is given by

$$\Delta \mathcal{E}_{t+1} = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} \mathcal{E}_t + \Sigma_{\mathcal{E}} \epsilon_{t+1}^{\mathbb{Q}}, \tag{4}$$

where $\epsilon_{t+1}^{\mathbb{Q}}$ is a Gaussian white noise. The n-period nominal yield is defined by

$$y_t^{(n)} \equiv -\frac{1}{n} \ln E_t^{Q}\left[\exp -\left\{\sum_{j=0}^{n-1} \left(\rho_0 + \rho_1' \mathcal{E}_{t+j}\right)\right\}\right],$$

[8] JPS find that yields span no more than 32% of their real activity measure and argue—as we do—that macro variables are not spanned by yields. They also find—as we do—that yields nearly span survey-based inflation forecasts, with R^2s up to 88% in their paper. They do not discuss whether expectations should be spanned in a term structure model.

and the following closed-form solution:

$$y_t^{(n)} = a_n + b_n' \mathcal{E}_t, \tag{5}$$

with the coefficients a_n and b_n given in Appendix A1. Equations (2)–(5) are standard: they postulate a set of yield factors $\mathcal{E}_t$ (Equation 2) driving the short rate (Equation 3) and with Markovian dynamics under $\mathbb{Q}$ (Equation 4) so that the solution for yields is affine (Equation 5).

Note that $\mathcal{E}_t$ plays two roles. It represents a set of risk factors driving yields and it represents the conditional mean of $\mathcal{Z}_t$. Forward-looking yields are consistent with long-run risk equilibrium models (e.g., Hasseltoft, 2012; Bansal and Shaliastovich, 2013). Forward-looking rules for the short rate are discussed in Ang, Dong, and Piazzesi (2007) in the context of affine term structure models. However, the distinction between rules that depend on $\mathcal{Z}_t$ and forward-looking rules that depend on $\mathcal{E}_t$ is blurred in Markovian models, since $\mathcal{E}_t$ is a function of $\mathcal{Z}_t$. This distinction plays an important role in what follows.

2.3 HISTORICAL DYNAMICS

The change of measure ξ_t is exponential-affine:

$$\xi_{t+1} = \frac{\exp(\lambda_t \epsilon_{t+1}^{\mathbb{Q}})}{E_t[\exp(\lambda_t \epsilon_{t+1}^{\mathbb{Q}})]}, \tag{6}$$

with the $\mathcal{N}_{\mathcal{Z}} \times 1$ vector of prices of risk λ_t given by

$$\lambda_t \equiv \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathcal{E}_t. \tag{7}$$

The change of measure ξ_t and the prices of risk λ_t are standard (Piazzesi, 2005), but with the difference that they are functions of $\mathcal{E}_t$ and not $\mathcal{Z}_t$: the risk premium is forward-looking. From standard results, the combination of this pricing kernel with Gaussian innovations in Equation (4) implies the following dynamics for $\mathcal{E}_t$ under the historical measure $\mathbb{P}$:

$$\Delta \mathcal{E}_{t+1} = K_0^{\mathbb{P}} + K_1^{\mathbb{P}} \mathcal{E}_t + \Sigma_{\mathcal{E}} \epsilon_{t+1}^{\mathbb{P}}, \tag{8}$$

with parameters given by

$$K_0^{\mathbb{P}} = K_0^{\mathbb{Q}} - \Sigma_{\mathcal{E}} \tilde{\lambda}_0 \tag{9}$$

$$K_1^{\mathbb{P}} = K_1^{\mathbb{Q}} - \Sigma_{\mathcal{E}} \tilde{\lambda}_1. \tag{10}$$

The dynamics for $\mathcal{E}_t$ require identification assumptions. These are provided in Proposition 1, which is an adaptation of Proposition 1 in JSZ.

Proposition 1

Every canonical CM-MTSM is observationally equivalent to a canonical CM-MTSM with $i_t = \iota \cdot \mathcal{E}_t$, where ι is a vector of ones, and

$$\Delta\mathcal{E}_{t+1} = K_0^{\mathbb{P}} + K_1^{\mathbb{P}}\mathcal{E}_t + \Sigma_{\mathcal{E}}\epsilon_{t+1}^{\mathbb{P}} \tag{11}$$

$$\Delta\mathcal{E}_{t+1} = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}}\mathcal{E}_t + \Sigma_{\mathcal{E}}\epsilon_{t+1}^{\mathbb{Q}}, \tag{12}$$

where $\epsilon_t^{\mathbb{P}}$ and $\epsilon_t^{\mathbb{Q}}$ are Gaussian white noise under $\mathbb{Q}$ and $\mathbb{P}$, respectively, and $K_1^{\mathbb{Q}}$ is an ordered real Jordan form, $K_{0,1}^{\mathbb{Q}} = k_{\infty}^{\mathbb{Q}}$, $K_{0,i}^{\mathbb{Q}} = 0$ for $i > 1$.

The proof follows the same steps as the proof for Proposition 1 in JSZ, but substituting $\mathcal{E}_t$ for their generic X_t.

2.4 EXPECTATIONS AND STATE VARIABLES

The yield factors $\mathcal{E}_t$ span the conditional expectations of $\mathcal{Z}_{t+1}$ by construction: $\mathcal{Z}_{t+1} = \mathcal{E}_t + u_{t+1}$ for some zero-mean unpredictable innovation u_{t+1}. To complete the dynamics of $\mathcal{Z}_{t+1}$, we assume that u_{t+1} corresponds to a rotation of the innovations in $\mathcal{E}_{t+1}$,

$$\mathcal{Z}_{t+1} = \mathcal{E}_t + \Sigma\epsilon_{t+1}^{\mathbb{P}}. \tag{13}$$

That is, $u_{t+1} \equiv \Sigma\epsilon_{t+1}^{\mathbb{P}}$ where Σ is lower diagonal and positive-definite. Substituting $\epsilon_{t+1}^{\mathbb{P}} = \Sigma^{-1}(\mathcal{Z}_{t+1} - \mathcal{E}_t)$ in Equation (8) and rearranging slightly, we have that

$$\mathcal{E}_{t+1} = K_0^{\mathbb{P}} + (I_{\mathcal{N}_{\mathcal{Z}}} + K_1^{\mathbb{P}})\mathcal{E}_t + \Sigma_{\mathcal{E}}\Sigma^{-1}(\mathcal{Z}_{t+1} - \mathcal{E}_t). \tag{14}$$

In other words, the conditional mean $\mathcal{E}_{t+1}$ at time $t+1$ is a combination of its previous value $\mathcal{E}_t$ with weight $I_{\mathcal{N}_{\mathcal{Z}}} + K_1^{\mathbb{P}}$ and today's surprise $\mathcal{Z}_{t+1} - \mathcal{E}_t$ with weight $\Sigma_{\mathcal{E}}\Sigma^{-1}$. One can check that the conditional mean of $\mathcal{Z}_{t+1}$ is given by

$$\Delta\mathcal{E}_{t+1} = K_0^{\mathbb{P}} + K_1^{\mathbb{P}}\mathcal{Z}_t, \tag{15}$$

and that $\mathcal{Z}_t$ is Markovian (i.e., a VAR(1)) if the following restriction holds:

$$\Sigma_{\mathcal{E}}\Sigma^{-1} = I_{\mathcal{N}_{\mathcal{Z}}} + K_1^{\mathbb{P}}. \tag{16}$$

The restriction in Equation (16) shows that a Markovian model mixes the previous value $\mathcal{E}_t$ and the surprise term $\mathcal{Z}_{t+1} - \mathcal{E}_t$ with equal weights to update the expectations $\mathcal{E}_{t+1}$.

The assumption that the same shocks drive $\mathcal{Z}_t$ and $\mathcal{E}_t$ is uncontroversial: it holds in a VAR(1) model (see Equation (15)). The only novel assumption is our departure from the Markovian restriction (16). Piazzesi and Schneider (2006) also feature a non-Markovian model with learning, and the conditional mean dynamics are similar to the state dynamics in some long-run risk models.[9] In addition, most dynamic stochastic general equilibrium (DSGE) models imply that endogenous state variables and their expectations share the same set of structural shocks.[10] Finally, a conditional mean dynamic model is closely related to the standard GARCH model where the squared innovations drive the conditional variance.

Importantly, $\mathcal{E}_t$ is not the conditional mean of $\mathcal{Z}_{t+1}$ under the risk-neutral measure $\mathbb{Q}$. This is given by

$$\mathcal{E}_t^{\mathbb{Q}} = \mathcal{E}_t + \Sigma\Sigma_{\mathcal{E}}^{-1}\left(K_0^{\mathbb{Q}} - K_0^{\mathbb{P}}\right) + \Sigma\Sigma_{\mathcal{E}}^{-1}\left(K_1^{\mathbb{Q}} - K_1^{\mathbb{P}}\right)\mathcal{E}_t \tag{17}$$

(see Appendix A2). Therefore, $\mathcal{E}_t^{\mathbb{Q}}$ is an affine transformation of $\mathcal{E}_t$ and inherits the Markov property under $\mathbb{Q}$. It follows directly from Equation (17) that the risk premium—the spread between $\mathcal{E}_t$ and $\mathcal{E}_t^{\mathbb{Q}}$—is given by

$$\mathcal{E}_t^{\mathbb{Q}} - \mathcal{E}_t = \bar{\lambda}_0 + \bar{\lambda}_1\mathcal{E}_t.$$

It follows that (i) the risk premium in a CM-MTSM depends on the entire history of $\mathcal{Z}_t$ via the recursion for $\mathcal{E}_t$ in Equation (8), and (ii) the risk premium is forward-looking.

2.5 FROM LATENT TO OBSERVABLE FACTORS

Observable variables may include macro variables $\mathcal{M}_t$ and yield factors $\mathcal{P}_t$. Take a number $0 \leq \mathcal{N}_M \leq \mathcal{N}_Z$ of macro variables $\mathcal{M}_t$ that are spanned by the state $\mathcal{Z}_t$,

$$\mathcal{M}_t = \gamma_0 + \gamma_1\mathcal{Z}_t, \tag{18}$$

[9] See, in particular, the NBER version of Bansal and Yaron (2004).

[10] Including cases where the linearized model has a VAR(1) representation. In addition, the state variables often do not have a finite-order autoregressive representation and, in that case, the solution to the DSGE has a recursive structure similar to Equation (8). See Fernandez-Villaverde, Rubio-Ramirez, and Sargent (2005), Ravenna (2007) and the references therein on the invertibility problem associated with finding a finite-order VAR representation in (linearized) DSGE models.

where γ_0 is a $\mathcal{N}_M \times 1$ vector and γ_1 is a $\mathcal{N}_M \times \mathcal{N}_Z$ rectangular matrix. In addition, take $\mathcal{N}_L = \mathcal{N}_Z - \mathcal{N}_M$ observed portfolios of yields $\mathcal{P}_t$ that are measured without errors,

$$\mathcal{P}_t = W y_t = a_W + b_W \mathcal{E}_t,$$

where y_t stacks the cross-section of $\mathcal{N} \geq \mathcal{N}_\mathcal{Z}$ individual yields given by Equation (5), W is a $\mathcal{N}_L \times \mathcal{N}$ matrix of weights, $a_W \equiv W[a_{n_1} \ldots a_{n_\mathcal{N}}]'$, and $b_W = W[b_{n_1} \ldots b_{n_\mathcal{N}}]'$. This framework can accommodate different numbers of observable macro variables $\mathcal{M}_t$ and yield portfolios $\mathcal{P}_t$. Proposition 2 provides the joint dynamics of the observable.

Proposition 2
The $\mathcal{N}_Z \times 1$ vector of observable $\mathcal{X}_t$,

$$\mathcal{X}_t \equiv \begin{bmatrix} \mathcal{M}_t \\ \mathcal{P}_t \end{bmatrix} = C + D_1 \mathcal{Z}_t + D_2 \mathcal{E}_t, \tag{19}$$

with C, D_1, and D_2 given by

$$C \equiv \begin{bmatrix} \gamma_0 \\ a_W \end{bmatrix} \qquad D_1 \equiv \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix} \qquad D_2 \equiv \begin{bmatrix} 0 \\ b'_W \end{bmatrix}, \tag{20}$$

follows a conditional mean model analogous to Equations (11)–(13) and (17).

The result follows from applying the mapping in Equation (19) to the dynamics of $\mathcal{Z}_t$. The mapping between parameters is provided in Appendix A3.

We can then use the observed macro variables $\mathcal{M}_t$ and yield portfolios $\mathcal{P}_t$ to obtain an equivalent CM model based on the observable $\mathcal{X}_t$ instead of the latent $\mathcal{Z}_t$. Combined with the model identification from Proposition 1, Proposition 2 gives rise to a canonical form similar to that of JLS but with one important distinction. In their Markovian setup, the state variables $\mathcal{Z}_t$ necessarily span yields and macro variables, while in our case there is a separation. The states $\mathcal{Z}_t$ span the macro variables $\mathcal{M}_t$ while $\mathcal{E}_t$ span the cross-section of yields. This separation is possible because $\mathcal{Z}_t$ is not Markovian—$\mathcal{E}_t$ is not a function of $\mathcal{Z}_t$. The following theorem formalizes the canonical form. Relative to JLS, the parameterization introduces the matrix Σ^*, with dimensions varying with the number of macro variables; its role is discussed in the following section.

Theorem 1
Canonical Form.

Suppose that the $\mathcal{N}_L$ portfolios of yields $\mathcal{P}_t$ and the $\mathcal{N}_M$ macro variables $\mathcal{M}_t$ are observed without errors, with $\mathcal{N}_L + \mathcal{N}_M = \mathcal{N}_Z$, as in Proposition 2.

Then any canonical CM-MTSM is equivalent to a unique canonical CM-MTSM whose state variables are $\mathcal{X}_t' = [\mathcal{M}_t\ \mathcal{P}_t']$. That is, the short rate is given by

$$i_t = \rho_{0X} + \rho_{1X}' \mathcal{E}^{\mathbb{P}}_{\mathcal{X},t}, \tag{21}$$

and the dynamics of $\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t} \equiv E_t[\mathcal{X}_{t+1}]$ under $\mathbb{Q}$ and $\mathbb{P}$ are given by

$$\begin{aligned} \Delta\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t+1} &= K^{\mathbb{Q}}_{0X} + K^{\mathbb{Q}}_{1X}\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t} + \Sigma_{\mathcal{E}_\mathcal{X}}\epsilon^{\mathbb{Q}}_{\mathcal{X},t+1} \\ \Delta\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t+1} &= K^{\mathbb{P}}_{0X} + K^{\mathbb{P}}_{1X}\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t} + \Sigma_{\mathcal{E}_\mathcal{X}}\epsilon^{\mathbb{P}}_{\mathcal{X},t+1}. \end{aligned} \tag{22}$$

The dynamics of $\mathcal{X}_{t+1}$ are given by

$$\begin{aligned} \mathcal{X}_{t+1} &= \mathcal{E}^{\mathbb{Q}}_{\mathcal{X},t} + \Sigma_X\epsilon^{\mathbb{Q}}_{\mathcal{X},t+1} \\ \mathcal{X}_{t+1} &= \mathcal{E}^{\mathbb{P}}_{\mathcal{X},t} + \Sigma_X\epsilon^{\mathbb{P}}_{\mathcal{X},t+1}, \end{aligned} \tag{23}$$

under $\mathbb{Q}$ and $\mathbb{P}$, respectively, and where the link between $\mathcal{E}^{\mathbb{Q}}_{\mathcal{X},t}$ and $\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t}$—the risk premium—is given by

$$\mathcal{E}^{\mathbb{Q}}_{\mathcal{X},t} = \mathcal{E}^{\mathbb{P}}_{\mathcal{X},t} + \Sigma_X\Sigma^{-1}_{\mathcal{E}_\mathcal{X}}\left(K^{\mathbb{Q}}_{0X} - K^{\mathbb{P}}_{0X}\right) + \Sigma_X\Sigma^{-1}_{\mathcal{E}_\mathcal{X}}\left(K^{\mathbb{Q}}_{1X} - K^{\mathbb{P}}_{1X}\right)\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t}. \tag{24}$$

The canonical form is parameterized by

$$\Theta^{\mathcal{X}} = \left(K^{\mathbb{P}}_{0X}, K^{\mathbb{P}}_{1X}, \lambda^{\mathbb{Q}}, k^{\mathbb{Q}}_{\infty}, \gamma_0, \gamma_1, \Sigma_{\mathcal{E}_\mathcal{X}}, \Sigma^*\right),$$

where Σ^* is a $\mathcal{N}_M \times \mathcal{N}_Z$ matrix (see Appendix A4 for details).

2.6 DISCUSSION

2.6.a *Dynamic MTSMs—Joslin, Le, and Singleton (2013a)*

Our canonical form nests the Markovian macro-finance DTSM in JLS whenever $\Sigma_{\mathcal{E}_\mathcal{X}} = (I_{\mathcal{N}_\mathcal{Z}} + K^{\mathbb{P}}_{1X})\Sigma_X$. As discussed in Section 2.4, this restriction implies that $\mathcal{X}_t$ is Markovian under $\mathbb{P}$ and, therefore, that $\mathcal{E}^{\mathbb{P}}_{\mathcal{X},t}$ and $\mathcal{E}^{\mathbb{Q}}_{\mathcal{X},t}$ are affine functions of $\mathcal{X}_t$ (see Theorem 1). In turn, this implies that $\mathcal{X}_{t+1}$ follows a Gaussian VAR(1) under $\mathbb{Q}$ and that both yields and macro variables are linear in $\mathcal{X}_t$. In this case, the matrix Σ^* drops from the parameterization. We can either estimate $\Sigma_{\mathcal{E}_\mathcal{X}}$ and compute $\Sigma_X = \Sigma_{\mathcal{E}_\mathcal{X}}(I_{\mathcal{N}_\mathcal{Z}} + K^{\mathbb{P}}_{1X})^{-1}$, as before, or we can estimate Σ_X directly, as in a VAR(1).

2.6.b. *DTSMs—Joslin, Singleton, and Zhu (2011)*

Our canonical model also nests the yield-only model in JSZ. Consider the expression for Σ_X in terms of Σ^* and other model parameters given in Appendix A4:

$$\Sigma_X = D_2 D^{-1} \Sigma_{\mathcal{E}_\mathcal{X}} + \begin{pmatrix} \Sigma^* \\ 0_{(\mathcal{N}_L \times \mathcal{N}_Z)} \end{pmatrix} \tag{25}$$

$$D \equiv D_1 + D_2(K_1^{\mathbb{P}} + I_{\mathcal{N}_Z}), \tag{26}$$

where D_1 and D_2 are given in Equation (20).[11] If $\mathcal{X}_t \equiv \mathcal{P}_t$, then Theorem 1 has that $\mathcal{N}_L = \mathcal{N}_Z$, $\Sigma^* = 0$, $D_1 = 0$ and $D_2 = b_W$. Then, Equation (25) yields $\Sigma_{\mathcal{E}_\mathcal{X}} = \Sigma_X + K_{1X}^{\mathbb{P}} \Sigma_X$, implying that $\mathcal{P}_t$ follows a VAR(1) under $\mathbb{Q}$ and $\mathbb{P}$, as in JSZ. Hence, there is no difference between standard and CM DTSMS in this specific case. Note that γ_0 and γ_1 are zero and drop from the parameterization.

Alternatively, considering cases beyond Theorem 1, one could substitute additional yield portfolios for the missing macro variables in Equation (18). Then, the yield portfolios have CM dynamics unless the Markovian restriction holds, which would yield again to JSZ. Note that Theorem 1 must be modified slightly (γ_0 and γ_1 would now link some yield portfolios to Z_t, but would not be free in that case).

2.6.c. *Dynamic macro-only term structure models*

The opposite case excludes yield factors and only uses observable macro variables, $\mathcal{X}_t \equiv \mathcal{M}_t$. Then, Σ^* is square (since $\mathcal{N}_M = \mathcal{N}_Z$), Σ_X can be estimated separately from $\Sigma_{\mathcal{E}_\mathcal{X}}$, and $\mathcal{X}_t$ has unrestricted conditional mean dynamics. Its $\mathbb{P}$-parameters can be estimated directly based on the macro variables. In addition, the cross-section of yields is driven entirely by the history of macro variables, and the risk-neutral parameters can be estimated using $\mathcal{E}_{\mathcal{X},t}$ filtered from the recursion in Equation (8). We do not consider this case in the empirical application. We focus on cases where $\mathcal{X}_t$ mixes macro and yield variables.

[11] The definition in Equation (26) involves parameters from the generic representation. An alternative definition based on the parameters of canonical representation in Theorem 1 is given by

$$\left(I_{\mathcal{N}_Z} \otimes D_1 + \left(I_{\mathcal{N}_Z} + K_{1X}^{\mathbb{P}'}\right) \otimes D_2\right) vec\left(D^{-1}\right) = vec\left(I_{\mathcal{N}_Z}\right),$$

which is well defined if the first term can be inverted (which is how we proceed, in practice).

2.6.d. *Survey-as-mean models—Chun (2011)*

Our approach is also closely related to models where survey forecasts are used as yield factors (Chun, 2011). But there are drawbacks to including survey inflation forecasts $\bar{\pi}_t^s$ in the observable $\mathcal{M}_t$ instead of in the underlying macro series. First, this approach assumes away expectation errors in survey forecasts. Second, survey participants do not report identical forecasts and it is not clear that the median response is consistent with the expectations implicit in bond prices. Third, Section 2.1 suggests that survey forecasts should not be included in the span of yields.

More importantly, using $\bar{\pi}_t^s$ instead of π_t, or surveys instead of macro variables, breaks the link between the conditional expectations and the observed macro series. Therefore, the model remains silent about the inflation-risk premium. However, our canonical form nests an extended version of Chun's 2011 approach where we include both the inflation rate and the survey inflation forecast. Once $\mathcal{M}_t$ combines π_t and $\bar{\pi}_t^s$, the challenge is to find the set of restrictions guaranteeing that $\bar{\pi}_t^s$ corresponds to the model conditional mean up to a measurement error.[12] Continuing with the example of inflation, we have that

$$\pi_t = \gamma_{\pi 0} + \gamma_\pi' \mathcal{Z}_t, \tag{27}$$

and its conditional mean is given by $\mathcal{E}_{\pi,t} = E_t[\pi_{t+1}] = \gamma_{\pi 0} + \gamma_\pi' E_t[\mathcal{Z}_{t+1}] = \gamma_{\pi 0} + \gamma_\pi' \mathcal{E}_t$. Then, $\bar{\pi}_t^s$ and $\mathcal{E}_{\pi,t}$ differ by some measurement errors η_t^s only if

$$\bar{\pi}_t^s = \gamma_{\pi 0} + \gamma_\pi' \mathcal{E}_t + \eta_t^s, \tag{28}$$

where (i) $E^{\mathbb{P}}[\eta_t^s] = 0$ and (ii) $\mathrm{cov}(\eta_t^s, \eta_{t+h}^s) = 0 \quad h > 0$. In turn, Equation (28) implies restrictions on parameters of the $\mathbb{P}$-dynamics. With no loss of generality, consider the case where π_t and $\bar{\pi}_t^s$ are ordered first and second in the vector $\mathcal{X}_t$, respectively. Then, the following condition

$$e_1' K_{0X}^{\mathbb{P}} = 0 \quad \text{and} \quad \delta \equiv e_2 - (I + K_{1X}^{\mathbb{P}})' e_1 = 0$$

implies that η_t^s is a measurement error (e_i has its i-th element equal to 1 and equal to 0 otherwise). In other words, $e_1' K_{0X}^{\mathbb{P}} = 0$ requires that the first element of the constant be zero and $\delta = 0$ requires that the second line of

[12] If survey data are used within a measurement equation, as we do in Section 3.2.3, there is no need to impose parametric restriction within the conditional mean specification (or within the nested VAR(1) specification).

the autoregressive matrix be equal to e_2'. Clearly, this condition can be implemented easily. It also has an intuitive economic interpretation:

$$\begin{aligned}\mathcal{E}_{\pi,t} &= e_1'\mathcal{E}_{\mathcal{X},t} = e_1'(K_{0X}^{\mathbb{P}} + (I + K_{1X}^{\mathbb{P}})\mathcal{E}_{\mathcal{X},t-1} + \Sigma_{\mathcal{E}_\mathcal{X}}\epsilon_t^{\mathbb{P}})\\ &= e_2'\mathcal{E}_{\mathcal{X},t-1} + e_1'\Sigma_{\mathcal{E}_\mathcal{X}}\epsilon_t^{\mathbb{P}}\\ &= e_2'\mathcal{X}_t + (e_1'\Sigma_{\mathcal{E}_\mathcal{X}} - e_2'\Sigma_X)\epsilon_t^{\mathbb{P}}\\ &= \bar{\pi}_t^s - e_1'\theta_X^{\mathbb{P}}\Sigma_X\epsilon_t^{\mathbb{P}},\end{aligned} \tag{29}$$

where $\theta_X^{\mathbb{P}} \equiv I + K_{1X}^{\mathbb{P}} - \Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}$. The model-implied forecast corresponds to the survey forecasts plus a rotation of innovations in $\mathcal{X}_t$ (including the innovations in survey forecasts). This guarantees that Equation (28) holds.[13] This simple condition, linking two strands of literature, only affects the likelihood of the inflation rate and it does not affect the fit of the remaining variables (relative to the unrestricted case). Of course, this link is also available in Markovian models (i.e., if $\theta_X^{\mathbb{P}} = 0$) but at the added cost that $\text{var}(\eta_t^s) = 0$. In other words, the model inflation forecast must correspond exactly to the survey forecast if the process is Markovian. Otherwise, Equation (28) cannot hold.

2.6.e. *VARMA representation*

The conditional mean representation was introduced by Fiorentini and Sentana (1998) in the context of time-series models, and differs significantly from the standard (VAR) representation used in macro-finance models. Nonetheless, the processes for $\mathcal{Z}_t$ and $\mathcal{X}_t$ have equivalent representations within the broader family of VARMA processes. For instance, combining the equations for $\mathcal{Z}_t$ and $\mathcal{E}_t$ together [i.e., Equations (8) and (13)] yields the following equivalent unrestricted VARMA(1,1) process:

$$\mathcal{Z}_{t+1} = K_0^{\mathbb{P}} + \phi^{\mathbb{P}}\mathcal{Z}_t - \theta^{\mathbb{P}}\Sigma\epsilon_t^{\mathbb{P}} + \Sigma\epsilon_{t+1}^{\mathbb{P}}, \tag{30}$$

where

$$\phi^{\mathbb{P}} = I_{\mathcal{N}_Z} + K_1^{\mathbb{P}} \qquad \theta^{\mathbb{P}} = \phi^{\mathbb{P}} - \Sigma_{\mathcal{E}}\Sigma^{-1}.$$

We can obtain similar representations for $\mathcal{Z}_t$ under $\mathbb{Q}$ and for $\mathcal{X}_t$ under each measure. Note that $\theta^{\mathbb{P}} = 0$ if and only if Equation (16) holds. In addition,

[13] The necessary and sufficient conditions for $\text{cov}(\eta_t^s, \eta_{t+h}^s) = 0 \quad h > 0$ is given in Equation (A.23) of Appendix A6 in the case with several macro and survey variables. This results makes clear that Equation (2.6.4) can hold for alternative sufficient conditions. In this section, we only consider one sufficient condition that impose no restrictions on the covariance matrix (which are not free in the canonical form).

note that $\theta_X^{\mathbb{P}}$ in the VARMA representation for $\mathcal{X}_t$ cannot be estimated separately from the other parameters (see Equation (25)). Any VARMA process has an equivalent VAR(1) representation, but with an extended state vector. Specifically, the $2\mathcal{N} \times 1$ extended vector $\widetilde{\mathcal{Z}}_t' \equiv [\mathcal{Z}_t' \; \mathcal{E}_t']$ follows the VAR(1) process:

$$\Delta \widetilde{\mathcal{Z}}_t = K_{0\tilde{Z}}^{\mathbb{Q}} + K_{1\tilde{Z}}^{\mathbb{Q}} \widetilde{\mathcal{Z}}_{t-1} + \Sigma_{\tilde{Z}} \epsilon_t^{\mathbb{Q}}, \tag{31}$$

with several cross-equation restrictions given by

$$K_{0\tilde{Z}}^{\mathbb{Q}} = \begin{bmatrix} 0_{\mathcal{N}_Z \times 1} \\ K_{0\tilde{Z}}^{\mathbb{Q}} \end{bmatrix}, \; K_{1\tilde{Z}}^{\mathbb{Q}} = \begin{bmatrix} -I_{\mathcal{N}_Z} & I_{\mathcal{N}_Z} \\ 0_{\mathcal{N}_Z \times \mathcal{N}_Z} & K_{1Z}^{\mathbb{Q}} \end{bmatrix}, \; \Sigma_{\tilde{Z}} = \begin{bmatrix} \Sigma & 0_{\mathcal{N}_Z \times \mathcal{N}_Z} \\ \Sigma_{\mathcal{E}} & 0_{N \times N} \end{bmatrix}.$$

2.6.f. *Using the extended VAR representation*

Given the extended VAR(1) representation, it may be tempting to apply the canonical form of JLS to the vector $\widetilde{\mathcal{Z}}_t$. Recall that JLS apply affine transformations to the latent VAR(1) process to achieve an observationally equivalent representation, but with a convenient parameterization. However, the argument applies only if the parameters $K_{0\tilde{Z}}^{\mathbb{Q}}$, $K_{1\tilde{Z}}^{\mathbb{Q}}$, and $\Sigma_{\tilde{Z}}$ are unrestricted. This is not the case here, and the cross-equation restrictions are the essence of our message. Those restrictions guarantee that one block of $\widetilde{\mathcal{Z}}_t$—the yield factors $\mathcal{E}_t$—spans another block of $\widetilde{\mathcal{Z}}_t$—the conditional expectation of $\mathcal{Z}_t$.

First, the restrictions on $K_{1\tilde{Z}}^{\mathbb{Q}}$ imply that the Jordan form $UK_{1\tilde{Z}}^{\mathbb{Q}}U^{-1}$ in JLS (for an appropriate matrix U) embodies several eigenvalue restrictions with no convenient expression. Second, the desired rotation $U\widetilde{\mathcal{Z}}_t U^{-1}$ is not arbitrary and depends on cross-equation restrictions. Third, the covariance matrix $\Sigma_{\tilde{Z}}$ does not have full rank, implying that the covariance matrix of the transformed vector does not have full rank and, therefore, that the likelihood function is singular and cannot be used for estimation unless we can deduce the appropriate dimension reduction. But this depends on the initial matrix $\Sigma_{\tilde{Z}}$, or else the dimension reduction is not observationally equivalent. In addition, the standard essentially affine price of risk specification, $\lambda_{\tilde{Z},t} = \lambda_{\tilde{Z}0} + \lambda_{\tilde{Z}1}\widetilde{\mathcal{Z}}_t$, is not identified. Heuristically, not all the blocks of the matrix $\lambda_{\tilde{Z}1}$ are identified, because there are only $\mathcal{N}_Z$ sources of risk.

3. Data and Estimation

We estimate different versions of CM-DTSMs based on USA and Canadian data separately. The objective is to highlight key differences between our approach and the standard Markovian model. For the USA, we estimate three cases, and the models are estimated based on different combinations of data. These include the term structure of nominal yields, of inflation swaps, and of inflation survey forecasts, to mitigate the effect of sampling uncertainty and to obtain better estimates of each component of nominal yields. For Canada, inflation swaps are unavailable, but we constrain the inflation Sharpe ratio to obtain a reliable decomposition of nominal yields.

3.1 DATA

The observable vector $\mathcal{X}_t$ includes two macro variables ($\mathcal{N}_M = 2$) and two yield portfolios: our specification has four latent factors ($\mathcal{N}_Z = 4$). Specifically, we use the monthly inflation rate $\pi_t \equiv \ln \frac{p_t}{p_{t-1}}$ and the unemployment rate g_t (i.e., $\mathcal{M}_t \equiv [\pi_t,\ g_t]$). The yield portfolios are the first two PCs of yields, labeled level and slope, respectively (i.e., $\mathcal{P}_t \equiv [l_t,\ s_t]$). The sampling frequency is monthly.

For the USA, we use data between January 1985 and December 2012. We use zero-coupon yields for maturities of 3 months, 6 months, and annually between 1 and 10 years, available from the Gurkaynak, Sack, and Wright (2006) data set available from the Federal Reserve Board. The unemployment rate and the inflation rate data are available from the Bureau of Labor Statistics.[14] We use the median of the inflation forecasts from the BC Financial Forecasts Survey. This survey is conducted monthly since 1985 and requests a forecast of the (annualized) CPI inflation rate for the current and future quarters up to five quarters ahead. Finally, we use zero-coupon inflation swap data with 2, 5, and 10 years to maturity, which are available from Bloomberg starting in March 2005.

For Canada, we use data between January 1992 and December 2012. Zero-coupon yields are available from the Bank of Canada. The unemployment rate and the inflation rate data are available from Statistics Canada. We use the median inflation forecasts from the survey of professional forecasters by Consensus Economics (CE).[15] In the last month of every quarter, the survey requests a forecast for each of the remaining quarters in the

[14] See http://www.federalreserve.gov/pubs/feds/2006/ for the yield data. We use the seasonally adjusted unemployment rate for ages 16 years and older from the Current Population Survey and the seasonally adjusted consumer price index (All Urban Consumers: All Items).

[15] We use the seasonally adjusted all-items consumer price index in Canada (StatCan Table 326-0020), and the seasonally adjusted Canadian unemployment rate (StatCan Table 282-0089). See http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/ for zero-coupon yield data. The CE survey is conducted in the second week of each month, but the inflation forecasts are updated only quarterly.

current calendar year, and for each quarter of the following calendar year. We use the first five quarters (longer-horizon forecasts are available only irregularly). Inflation swap data are not available in Canada and outstanding real bonds have very long maturities throughout the sample.

3.2 LIKELIHOOD

We fix the parameters controlling the unconditional means of the state variables to their respective sample averages. We stack the remaining parameters in the vector Ξ. Our estimator of Ξ is based on the joint (log) likelihood of $\mathcal{X}_t$, of the cross-section of yields y_t, the cross-section of survey inflation forecast $\bar{\pi}_t^s$, and the cross-section of inflation swap rates isr_t. We impose the usual stationarity conditions on $K_1^{\mathbb{P}}$ and $K_1^{\mathbb{Q}}$. We detail each component of the likelihood, corresponding to different types of measurement equations.

3.2.a. *State dynamics*

The first component of the joint likelihood is the conditional likelihood of $\mathcal{X}_t$:

$$l(\mathcal{X}_t|\mathcal{X}_{t-1};\,\Xi) = -\frac{1}{2}\left\{\frac{\mathcal{N}_{\mathcal{Z}}}{2}\ln(2\pi) + \ln(\det(\Sigma_X\Sigma_X')) + \left(\mathcal{X}_t - \mathcal{E}_{\mathcal{X},t-1}\right)'\left(\Sigma_X\Sigma_X'\right)^{-1}\left(\mathcal{X}_t - \mathcal{E}_{\mathcal{X},t-1}\right)\right\}, \tag{32}$$

where $\mathcal{E}_{\mathcal{X},t}$ is given by the recursion in Equation (22) with the initial value $\mathcal{E}_{\mathcal{X},0} = E[\mathcal{X}_t]$ and using the fact that $\epsilon^{\mathbb{P}}_{\mathcal{X},t+1} = \Sigma^{-1}\left(\mathcal{X}_{t+1} - \mathcal{E}_{\mathcal{X},t}\right)$.

3.2.b. *Yields*

The second component is the conditional likelihood of nominal yields. The observable $\mathcal{X}_t$ includes the first two PCs of yields measured without errors. The remaining PCs of yields, with the $\mathcal{N}_J \times \mathcal{N}_e$ loadings W^e, are stacked within a vector $\mathcal{P}_t^e$:

$$\mathcal{P}_t^e \equiv W^e y_t = a_e + b_e\mathcal{E}_{\mathcal{X},t} + \eta_t^e,$$

where $a_e \equiv W^e[a_{n_1}\ldots a_{n_{\mathcal{N}}}]'$, $b_e = W^e[b_{n_1}\ldots b_{n_{\mathcal{N}}}]'$, and η_t^e stacks Gaussian measurement errors with variance $\sigma^2_{e,n}$. The conditional log-likelihood of $\mathcal{P}_t^e$ is then given by

$$l\left(\mathcal{P}_t^e|\mathcal{X}_t;\,\Xi\right) = \sum_{n=1}^{\mathcal{N}_e}\left(-\frac{1}{2}\left\{\ln\left(2\pi\sigma^2_{e,n}\right) + \left(\frac{\eta^e_{t,n}}{\sigma_{e,n}}\right)^2\right\}\right). \tag{33}$$

3.2.c. *Surveys of inflation forecasts*

The third component is the likelihood of the survey data. For the USA, the BC Financial Forecasts for the (annualized) CPI inflation rate over a given quarter:

$$\bar{\pi}^{s}_{t,h} = \frac{12}{3} E^{s}_{t}\left[\sum_{j=h-2}^{h} \pi_{t+j}\right], \tag{34}$$

where h is the horizon of that quarter's last month. In the case of Canada, the survey asks for a forecast of the average of year-over-year inflation rates across all months in a given quarter. This is given by

$$\bar{\pi}^{s}_{t,h} = \frac{1}{3} E^{s}_{t}\left(\sum_{j=h-11}^{h} \left(\pi_{t+j} + \pi_{t+j+1} + \pi_{t+j+2}\right)\right). \tag{35}$$

In both countries, we use survey forecasts for the current quarter and the following four quarters.

We assume that any survey inflation forecast matches the model forecast up to some i.i.d. $\eta^{s}_{t,h} \sim N(0, \sigma^2_{s,h})$, which may capture measurement errors or forecasting errors by survey participants. Then, we compute the expectation within the model, leading to a measurement equation of the following form:

$$\bar{\pi}^{s}_{t,h} = a_{\pi,h} + b'_{\pi,h}\mathcal{E}_{\mathcal{X},t} + \eta^{s}_{t,h}, \tag{36}$$

where the coefficients $a_{\pi,h}$ and $b_{\pi,h}$ follow from the $\mathbb{P}$-dynamics in Equation (22). For US surveys, the coefficients are given by

$$\begin{aligned} a_{\pi,h} &= (b_{\pi,h} - e_1)'(K^{\mathbb{P}}_{1X})^{-1}K^{\mathbb{P}}_{0X} \\ b'_{\pi,h} &= \frac{1}{3}e'_1\left((I + K^{\mathbb{P}}_{1X})^{h-1} + (I + K^{\mathbb{P}}_{1X})^{h-2} + (I + K^{\mathbb{P}}_{1X})^{h-3}\right). \end{aligned} \tag{37}$$

The log-likelihood of the survey data is given by

$$l\left(\bar{\pi}^{s}_{t,h}|\mathcal{X}_t; \Xi\right) = -\frac{1}{2}\left\{\ln\left(2\pi\sigma^2_{s,h}\right) + \left(\frac{\eta^{s}_{t,h}}{\sigma_{s,h}}\right)^2\right\}, \tag{38}$$

when survey data are observed and zero otherwise (survey data are quarterly in Canada).

3.2.d. *Inflation swap rate*

The final component is the likelihood of the zero-coupon inflation swap. The swap contract specifies a fixed payment based on the rate $isr^{(m)}_t$ known today but paid m years in the future, and a floating payment contingent on the inflation rate realization $\pi_{t+1} + \ldots + \pi_{t+12m}$. There is no cash exchange at initiation. To preclude the absence of arbitrage, the fixed rate is given by

$$\begin{aligned} isr_t^{(m)} &= \frac{1}{m} E_t^{\mathbb{Q}}\left[\sum_{j=1}^{12m} \pi_{t+j}\right] + \eta_{t,m}^{isr} \\ &= a_{isr,m} + b'_{isr,m} \mathcal{E}_{\mathcal{X},t} + \eta_{t,m}^{isr}, \end{aligned} \tag{39}$$

where the coefficients $a_{isr,h}$ and $b_{isr,h}$ follow from the $\mathbb{Q}$-dynamics in Equation (22) and with the inflation swap measurement error $\eta_{t,m}^{isr} \sim N(0, \sigma_{isr,h}^2)$. The log-likelihood of the survey data is then given by

$$l\left(isr_t^{(m)} | \mathcal{X}_t; \Xi\right) = -\frac{1}{2}\left\{\ln\left(2\pi\sigma_{isr,m}^2\right) + \left(\frac{\eta_{t,m}^{isr}}{\sigma_{isr,m}}\right)^2\right\}. \tag{40}$$

3.2.e. *Joint estimation versus two-step estimation*

We cannot neatly separate the parameters affecting the conditional dynamics of $\mathcal{X}_t$ from those parameters governing the cross-section of yields and inflation swaps (under the risk-neutral measure). The covariance matrix $\Sigma_{\mathcal{E}_\mathcal{X}}$ is the same under each measure. As shown in JSZ, the remaining parameters that control the conditional dynamics can be estimated consistently in the standard Markovian case. This is not the case here. A simple count of parameters in $\bar{\lambda}_0$ and $\bar{\lambda}_1$ shows that we cannot freely shift the remaining parameters. In fact, Equation (25) shows that Σ_X is not separately identified from $\Sigma_{\mathcal{E}_\mathcal{X}}$ and that the interaction depends on the parameters from the $\mathbb{Q}$-dynamics via the matrix D and D_2 (Appendix A4 details the parameterization from Theorem 1). In other words, parameters from the $\mathbb{Q}$-dynamics show up in the $\mathbb{P}$-dynamics. Conversely, the asset pricing implications depend on the parameters of the $\mathbb{P}$-dynamics. The estimates of $\widehat{\mathcal{E}}_{\mathcal{X},t}$ drive asset prices but depend on $\mathbb{P}$-parameters (via the recursion in Equation (24)). This contrasts with the standard model, where the two dynamics can be estimated separately unless we impose additional restrictions on the prices of risk.

3.2.f. *Sharpe ratio constraint*

Inflation swap rates and real bonds are not reliably available in Canada. Therefore, the inflation-risk premium will be imprecisely estimated if the Sharpe ratio is left unrestricted (Duffee, 2010). The one-period-ahead Sharpe ratio associated with inflation risk,

$$\mathrm{SR}_{t,1}^{\pi} \equiv \sigma_\pi^{-1}\left(E_t^{\mathbb{Q}}[\pi_{t+1}] - E_t^{\mathbb{P}}[\pi_{t+1}]\right), \tag{41}$$

where $\sigma_\pi^2 \equiv \mathrm{Var}_t[\pi_{t+1}]$. This definition corresponds to Fisher's decomposition, but where the expectation $E_t^Q[\pi_{t+1}]$ is taken under the risk-neutral measure. Using Equation (24), the real short rate can be written analogously to the nominal rate, and is given by the first element of $\sigma_\pi^{-1}(\bar{\lambda}_0 + \bar{\lambda}_1 \mathcal{E}_{\mathcal{X},t})$. The first term in the numerator is the price of a contract that pays the realized inflation rate. The second term is the investor's expected inflation. Hence, the numerator is the inflation premium—the premium to investors for entering a fair bet on inflation. The ratio $\mathrm{SR}_{t,1}^\pi$ is a price per unit of volatility.

We constrain $\mathrm{SR}_{t,1}^\pi$ within a plausible range with a very high probability. The population distribution of $\mathrm{SR}_{t,1}^\pi$ is Gaussian with mean and variance $E[\mathrm{SR}_{t,1}^\pi] = \sigma_\pi^{-1}\left(\bar{\lambda}_{0,\pi} + \bar{\lambda}'_{1,\pi} E[\mathcal{E}_{\mathcal{X},t}]\right)$ and $\mathrm{Var}[\mathrm{SR}_{t,1}^\pi] = \sigma_\pi^{-2} \bar{\lambda}'_{1,\pi} \mathrm{Var}(\mathcal{E}_{\mathcal{X},t}) \bar{\lambda}_{1,\pi}$, respectively. We impose that

$$-\kappa \leq E[\mathrm{SR}_{t,1}^\pi] - 1.96\mathrm{Var}[\mathrm{SR}_{t,1}^\pi]^{1/2} \leq E[\mathrm{SR}_{t,1}^\pi] + 1.96\mathrm{Var}[\mathrm{SR}_{t,1}^\pi]^{1/2} \leq \kappa, \tag{42}$$

with $\kappa \geq 0$, implying a probability of 5% that $\mathrm{SR}_{t,1}^\pi$ is outside of the range $[-\kappa, \kappa]$ in population. We report results for $\kappa = 0.2$.[16] Our approach is consistent with Duffee (2010), who constrains the sample mean of the Sharpe ratio (but also suggests restricting the population distribution directly), and with Chernov and Mueller (2011), who penalize the excessive variability of the term premium.

3.3 MODELS

We estimate DTSMs based on the observable state variables $\mathcal{X}_t$ and where nominal yields, inflation survey forecasts and inflation swaps are driven by the conditional means $\mathcal{E}_t$. In each case, we estimate the parameters of the canonical form in Theorem 1. We estimate the unrestricted Conditional Mean model (CM) and the restricted VAR model (of order 1). The VAR is nested in Theorem 1 with the following parametric restriction:

$$\Sigma_{\mathcal{E}_\mathcal{X}} = (I_{\mathcal{N}_\mathcal{Z}} + K_{1X}^{\mathbb{P}})\Sigma_X,$$

implying that Σ^* drops from the parameterization, and that we can estimate Σ_X instead of $\Sigma_{\mathcal{E}_\mathcal{X}}$ (see Section 2.6.2). The benchmark CM model has four factors and forty-nine parameters.

For the USA, we consider three cases, where we vary the set of measurement equations used to estimate the models. Case A uses all measurement

[16] We also assess the effect of this restriction, and reestimate the model while varying κ from zero to infinity (the unconstrained case). Results are available from the authors.

Table II. Measurement errors—US yields

Standard deviation of measurement errors in annualized percentage for US yields. Cases A–C include different measurement equations for estimation. Case A includes measurement equations for the yields, the inflation survey forecasts, and the inflation swaps. Case B excludes the inflation swaps. Case C also excludes the inflation survey forecasts. Panels A and B report results for CM models and for VAR models, respectively. Monthly data 12/1984 to 12/2012.

Panel A: yields RMSEs in CM models												
Case	3	6	12	24	36	48	60	72	84	96	108	120
A	0.16	0.07	0.07	0.11	0.12	0.10	0.07	0.03	0.02	0.06	0.09	0.11
B	0.17	0.08	0.07	0.12	0.13	0.11	0.08	0.04	0.03	0.06	0.10	0.13
C	0.17	0.08	0.07	0.12	0.13	0.11	0.08	0.04	0.03	0.06	0.10	0.13
Panel B: yields RMSEs in VAR models												
Case	3	6	12	24	36	48	60	72	84	96	108	120
A	0.24	0.11	0.08	0.17	0.18	0.14	0.09	0.04	0.04	0.08	0.13	0.16
B	0.23	0.11	0.08	0.17	0.18	0.14	0.09	0.03	0.04	0.08	0.12	0.16
C	0.23	0.11	0.08	0.17	0.18	0.14	0.09	0.03	0.04	0.08	0.12	0.16

equations, combining yield, inflation swap, and inflation survey forecast data. Case B excludes the inflation swap data, while Case C also excludes the inflation survey forecast data. We reestimate the VAR and CM model for each case, assessing the contribution of different information sets and, more importantly, how each model's implications change across information set.

4. Results

4.1 FITTING THE CROSS-SECTION OF YIELDS, SURVEYS, AND INFLATION SWAPS

In Case A, the joint (log-) likelihood of the data at the estimated parameters increases by 1287 between the VAR and the CM model, with a corresponding increase of nine parameters. We can use a likelihood ratio to construct test-statistics for the null hypothesis that the Markovian restriction holds in the data. Its p-value is essentially zero in every case. Table II reports the RMSE of measurement errors for US yields for Cases A–C. Comparing the fit of yields across models, the results show that the CM models offer an improvement in every case, by more than 25% in some case. Comparing the fit across cases for a given model, the results are identical for

Table III. Measurement errors—US inflation swap and survey forecasts

Standard deviation of measurement errors in annualized percentage for US inflation swaps and BC inflation forecasts. Panels B and A report survey RMSEs and swap RMSEs, respectively. Monthly data 12/1984 to 12/2012.

Panel A : inflation RMSEs				Panel B: surveys RMSEs					
	2	**5**	**10**		**Q1**	**Q2**	**Q3**	**Q4**	**Q5**
VAR	1.12	0.54	0.34	VAR	0.48	0.44	0.45	0.49	0.27
CM	0.69	0.35	0.21	CM	0.55	0.45	0.46	0.50	0.23

the VAR models. The fit does not improve from Cases A to C, since we exclude data from the likelihood. The results are very similar across cases for the CM model as well. If anything, the fit of yields is best when inflation swap data and inflation survey forecast data are included in the likelihood.[17]

Table III reports the RMSE of measurement errors for US inflation swaps and survey inflation forecasts. Panel A reports the RMSEs for inflation swaps. The differences between models are large, especially for the 2-year inflation swap rate, where the RMSE is 1.12% and 0.80% for the VAR and the CM model, respectively. Panel B reports the survey RMSEs. The fit of survey forecasts is close. Finally, Table IV reports yields and inflation survey forecast RMSEs. The results are consistent: the CM model improves the fit of yields substantially, with no increase in the number of yield factors.

4.2 YIELDS LOADINGS

How does the CM model differ from the VAR in fitting yields? The answer lies in its spanning property. Figure 1A reports the estimated loadings of yields (including the short rate) on the risk factor $\mathcal{E}_{\mathcal{X},t}$ in the USA. The left panel shows the loadings on the inflation conditional mean, and the right panel shows the loadings on the unemployment rate conditional mean. These loadings are essentially zero for the VAR. This result is consistent with model $T3$ in Chun (2011), combining two yield factors and inflation survey forecasts. He finds small loadings on expected inflation. In other words, the conditional means of macro variables are unspanned by nominal yields in these models. This was not preordained: we did not impose spanning restrictions on the conditional means at estimation. For

[17] In contrast, Chernov and Mueller (2011) mention (but do not report results) that four-factor models have trouble matching yield and survey data, with yield errors increasing three- to five-fold. The need to include an additional factor that may arise because of their inclusion of long-horizon survey forecasts, or the much longer sample period.

Table IV. Measurement errors—Canadian data

Standard deviation of measurement errors in annualized percentage for Canadian yields and CE inflation forecasts. Panels A and B report yields RMSEs and survey RMSEs, respectively. The Sharpe ratio constraint is set to $\kappa = 0.2$ in every case. Monthly macro and yield data 01/1984 to 12/2012. Quarterly survey data Q1-1992/Q4-2012.

Panel A: yields RMSEs

	Maturity										
	3	6	12	24	36	48	60	84	96	108	120
VAR	0.28	0.09	0.11	0.17	0.14	0.09	0.05	0.09	0.11	0.13	0.14
CM	0.18	0.07	0.07	0.12	0.11	0.08	0.05	0.10	0.09	0.10	0.11

Panel B: surveys RMSEs

	Horizon				
	Q1	Q2	Q3	Q4	Q5
VAR	0.52	0.59	0.50	0.33	0.29
CM	0.53	0.59	0.51	0.32	0.28

the VAR model, this is not unexpected either. Section 2.1 shows that much of the variation in the macro variables is unrelated to variations in the yields.

The CM model paints a very different picture. The short-rate coefficients for the expected inflation and expected unemployment are 1.16 and −0.36, respectively. The pattern of loadings across maturities indicates that the level of yield rises but that its slope flattens when expected inflation increases. The loading on expected inflation decreases to 0.61 for the 10-year bond. Similarly, the pattern of loadings indicates that the level increases and the slope flattens when expected unemployment decreases. Panel B shows that the loadings for Canadian yields exhibit the same pattern. If anything, the influence of macro variables on yields is larger in the Canadian data, yet the yield loadings on the macro conditional means remain close to zero in the VAR.

4.3 FORECASTING INFLATION

Figure 2 shows the expected inflation at horizons of 3 months, 1 year, and 2 years from the CM model. Figure 2A and B reports the results for the USA and for Canada, respectively. Expected inflation is strongly procyclical and declines from 3% early in the sample to around 2% in 2004 in the USA and in 1994 for Canada. Figure 2 also shows that the slope of expected inflation

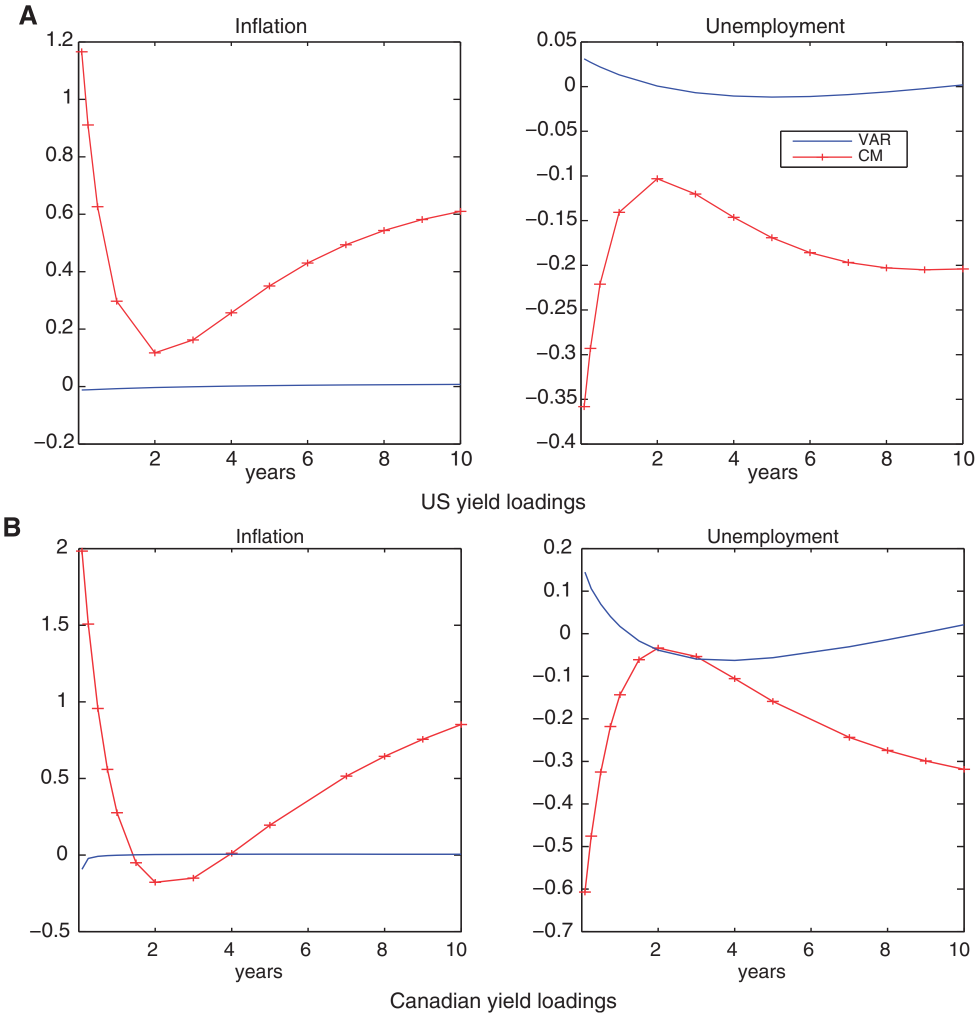


Figure 1. Nominal Yield Loadings. Loadings of nominal yields on the conditional mean of inflation and on the conditional mean of unemployment from the benchmark CM model in the US (Panel A) and in Canada (Panel B).

changes sign over time. Short-run inflation expectations stand above long-run inflation expectations when unemployment is high. The slope has been negative in the USA since 2008.

Yields should span conditional means in CM models. This requires that forecasts from the model should be as good as any other forecasts. Ang, Bekaert, and Wei (2007) argue that surveys provide the most accurate

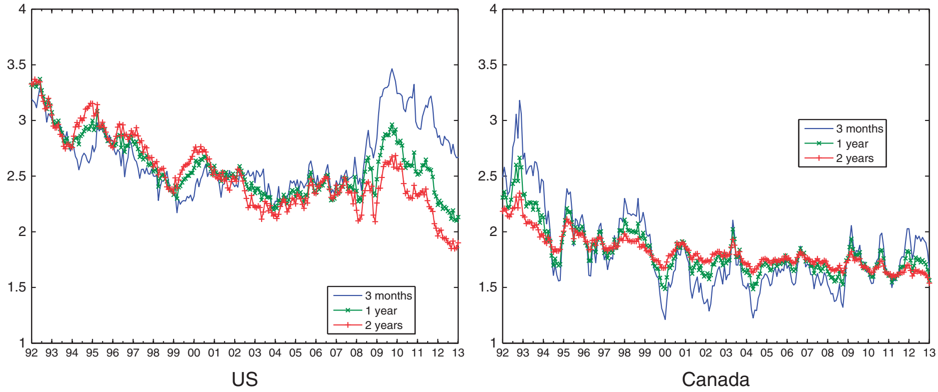


Figure 2. Term structure of expected inflation. (A) USA; (B) Canada. Expected inflation from the CM benchmark model, over the horizons 3 months, 1 year, and 2 years in the USA (A) and in Canada (B). Monthly data from January 1992 to December 2012.

inflation forecasts (see also the survey by Faust and Wright, 2011). We check that the CM model provides valid conditional expectations, using survey forecasts as a benchmark. We first estimate the models with data until December 1991, producing inflation forecasts that correspond to each survey forecast at the end of that month. For this purpose, we extend the sample back to 1985 in Canada. We then add 1 month to the estimation sample, produce new inflation forecasts, and repeat until the end of 2012. At each date, we also produce forecasts of the annualized inflation rate for horizons between 1 and 24 months ahead.

4.3.a. *Matching survey forecast accuracy*

Table V reports the ratio of the inflation forecast RMSEs from the models relative to the forecast RMSEs from the survey. Panels A and B report results for US and Canadian inflation forecasts, respectively. No model systematically improves upon the accuracy of survey forecasts, but most models perform reasonably well. In contrast to Ang, Bekaert, and Wei (2007), we find that the VAR and CM models match the accuracy of surveys, confirming the spanning assumption. The pattern is consistent across countries. While each model offers similar forecast accuracy when survey data are available, we show below that forecasts from the CM are superior when survey data are unavailable. Finally, note that conditional mean spanning requires ratios close to one but not necessarily equal to one. Survey participants and bond investors do not have and do not report the same conditional forecasts for future inflation.

Table V. Can model forecasts improve survey forecasts?

Ratio of inflation out-of-sample forecast RMSEs from the benchmark CM model relative to the RMSEs from survey forecasts. Monthly forecast horizons. Panel A shows the results for US inflation where we use monthly BC survey forecasts. Panel B shows the results for Canadian inflation where we use quarterly CE survey forecasts. The out-of-sample data start in December 1991 and end in December 2012.

Panel A: US inflation forecasts						Panel B: Canadian inflation forecasts					
	Q1	**Q2**	**Q3**	**Q4**	**Q5**		**Q1**	**Q2**	**Q3**	**Q4**	**Q5**
VAR	1.01	1.02	1.02	1.03	1.00	VAR	0.99	1.04	0.95	1.05	1.04
CM	0.99	1.00	1.01	1.02	1.00	CM	1.00	1.03	0.93	0.99	1.02

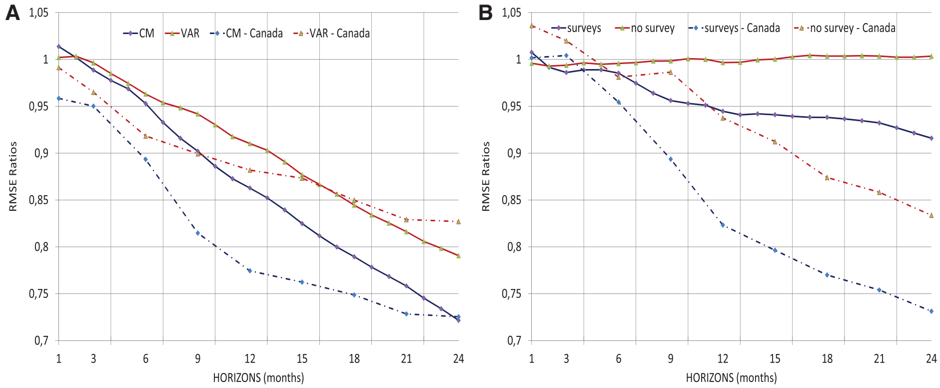
4.3.b. *Using survey data at estimation*

The model-implied inflation forecasts are accurate because estimation uses the likelihood of survey inflation forecasts. Figure 3A reports the ratios of inflation forecast RMSEs from models estimated using the full likelihood relative to that from models estimated excluding the likelihood of survey data (Case A relative to Case C). Figure 3A reports RMSE ratios for US and Canadian data. The impact of the survey data is apparent. The forecast RMSEs deteriorate by 5–10% for 1-year forecasts in US data, and by as much as 20–25% for 2-year forecasts. The effect is typically larger in Canadian data. The deterioration is more severe for the CM models, except in Canadian data for the longer horizons. This pattern reflects this model's larger parameterization. The likelihood of survey data plays a more important role as the number of parameters increases, in effect mitigating the loss in parsimony. To summarize, estimates that neglect survey data suffer from substantial bias and small-sample sampling errors. Adding surveys to the likelihood effectively lengthens the sample (Kim, 2007; Kim and Orphanides, 2012). Ang, Bekaert, and Wei (2007) find that survey forecasts are more accurate than model-based forecasts, but they do not use survey data at estimation.[18]

4.3.c. *Using the Markovian restriction*

The Markovian restriction makes inflation forecasts less accurate. Figure 3B reports the ratio of forecast RMSEs from CM models relative to that from

[18] It is essential to include yield factors in state $\mathcal{X}_t$ to match the accuracy of survey inflation forecasts. Results are available from the authors.



With or without surveys $\qquad$ CM vs VAR

Figure 3. Out-of-sample inflation forecast. Ratios of forecast RMSEs. (A) Reports the RMSEs for a given model estimated with survey data relative to the same model estimated without survey data (Case A versus Case C). Results are reported for the CM and the VAR models, both for USA and Canadian data. (B) Reports the RMSEs for the CM model relative to the corresponding VAR model using all data. Results are reported cases with and without survey data (Case A and Case C), for US data and Canadian data, respectively. Monthly data. Estimation sample: 1985–92. Out-of-sample: 1992–2012.

VAR models. Figure 3B reports results for the USA and Canada, using all the available data or excluding the survey data (Cases A and B). For the USA, the CM model yields accuracy improvements of 5% at the 1-year horizon and nearly 10% at the 2-year horizon (significant at the 1% level), but the CM model yields no improvement in Case B. In Canada, the improvements are remarkable. The CM model produces accuracy gains in the 20–25% range at the 1-year and 2-year horizons.

How can every model perform similarly when compared to survey forecasts (Table V), while the CM model outperforms when compared to the VAR? First, survey forecasts are available only quarterly in Canada, and the comparison with model forecasts can be done only quarterly. However, the comparison between models is done with monthly data. We find that forecast errors from the VAR models are also more serially correlated: this model performs relatively poorly in months when survey forecasts are not available. But survey forecasts are available monthly in the USA. The forecasting horizons also drive the contrasting results. The comparison with survey forecasts involves forecasts for inflation in specific quarters. On the other hand, the comparison between models involves forecasts of total inflation between now and some future horizon h-months ahead. These horizons are relevant, since they are used in the yield decomposition below. The RMSEs between these two horizons can only differ because of the covariance between the forecast errors across quarters. Indeed, we find that forecast errors from the VAR models are more correlated across horizons (unreported).

4.4 WHY CONDITIONAL MEAN MODELS FOR INFLATION FORECASTS?

How does a conditional mean model offer more accurate forecasts? To answer this question, consider the VAR model,

$$\begin{aligned} \text{VAR}: \quad \mathcal{E}_{\mathcal{X},t} &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{X}_t \\ &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{E}_{\mathcal{X},t-1} + (K_{1X} + I_{\mathcal{N}_Z})(\mathcal{X}_t - \mathcal{E}_{\mathcal{X},t-1}) \\ &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{E}_{\mathcal{X},t-1} + (K_{1X} + I_{\mathcal{N}_Z})u_{\mathcal{X},t}, \end{aligned} \tag{43}$$

with $u_{\mathcal{X},t} \equiv (\mathcal{X}_t - \mathcal{E}_{\mathcal{X},t-1})$, and contrast with the CM model,

$$\begin{aligned} \text{CM}: \quad \mathcal{E}_{\mathcal{X},t} &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{E}_{\mathcal{X},t-1} + \Sigma_{\mathcal{E}_\mathcal{X}}\epsilon_t^{\mathbb{P}} \\ &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{E}_{\mathcal{X},t-1} + \Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}(\mathcal{X}_t - \mathcal{E}_{\mathcal{X},t-1}) \\ &= K_{0X} + (K_{1X} + I_{\mathcal{N}_Z})\mathcal{E}_{\mathcal{X},t-1} + \Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}u_{\mathcal{X},t}, \end{aligned} \tag{44}$$

where the weights $(K_{1X} + I_{\mathcal{N}_Z})$ on the prior expectations $\mathcal{E}_{\mathcal{X},t-1}$ can differ from the weights $\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}$ on the innovations $u_{\mathcal{X},t}$. Tables VI–VII report parameter estimates for the USA and for Canada, respectively. In each case, Panel A reports estimates of the matrices $K_{1X} + I_{\mathcal{N}_Z}$ and $\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}$. Panel B reports estimates of the standard deviations (on the diagonals) and the correlations (off the diagonals) implied from the estimate of the covariance matrix $\Sigma_X\Sigma_X'$ and $\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_{\mathcal{E}_\mathcal{X}}'$.

Focusing on the first line of $(K_{1X} + I_{\mathcal{N}_Z})$ and $\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}$, the estimated dynamics of inflation expectations $\mathcal{E}_{\pi,t} = E_t[\pi_{t+1}]$ illustrate the difference:

$$\begin{aligned} \mathcal{E}_{\pi,t} &= \underset{(0.03)}{0.98} \times \mathcal{E}_{\pi,t-1} - \underset{(0.001)}{0.01} \times u_{\pi,t} - \underset{(0.02)}{0.04} \times \mathcal{E}_{g,t-1} + \underset{(0.06)}{0.51} \times u_{g,t} \\ &\quad + \underset{(0.003)}{0.008} \times \mathcal{E}_{l,t-1} + \underset{(0.01)}{0.01} \times u_{l,t} - \underset{(0.02)}{0.02} \times \mathcal{E}_{s,t-1} + \underset{(0.04)}{0.02} \times u_{s,t}, \end{aligned} \tag{45}$$

and, again, we have very similar results in the case of Canada:

$$\begin{aligned}\mathcal{E}_{\pi,t} = \underset{(0.15)}{1.03} \times \mathcal{E}_{\pi,t-1} - \underset{(0.002)}{0.005} \times u_{\pi,t} - \underset{(0.06)}{0.07} \times \mathcal{E}_{g,t-1} + \underset{(0.07)}{0.35} \times u_{g,t} \\ + \underset{(0.003)}{0.003} \times \mathcal{E}_{l,t-1} - \underset{(0.01)}{0.03} \times u_{l,t} - \underset{(0.04)}{0.04} \times \mathcal{E}_{s,t-1} + \underset{(0.06)}{0.26} \times u_{s,t},\end{aligned} \tag{46}$$

with standard errors provided in parentheses.[19]

First, expected inflation is very persistent in both countries: inflation innovations are transitory and have close to no effect on the expectation update. Also, note from Tables VI and VII that inflation innovations have no significant effect on $\mathcal{E}_{l,t}$ and $\mathcal{E}_{s,t}$, while their effect on $\mathcal{E}_{g,t}$ is small (see the first column of $\Sigma_{\mathcal{E}_{\mathcal{X}}}\Sigma_X^{-1}$). Hence, the intuition from Kim (2007) that inflation combines a persistent conditional mean component with transitory noise (i.e., an ARMA(1,1) process) carries over in our multivariate context. The VAR cannot distinguish between these two components.

Second, updates of expected inflation are driven by innovations in the other variables since the coefficient on inflation innovation is almost zero (albeit significant in Canada). The CM model uses the span of $u_{\mathcal{X},t}$ to capture predictable variations in expected inflation. Multiplying coefficients by the corresponding standard deviation shows that unemployment innovations are the most significant economically (standard deviation of 0.26). Slope innovations also play an important role in Canada (standard deviation of 0.45). In contrast, unemployment and slope innovations are only weakly connected to expected inflation updates in the VAR model (unreported).

Third, the estimated weights on the prior expectations differ from the estimated weights on the innovations in every case, and with opposite signs in every case. Expected inflation increases with a higher expected level of the yield curve, but shows no response to surprise changes in the level. Expected inflation increases with an expected rise in the slope, but decreases with a surprise tightening of the short rate. An increase of s_t lifts short-maturity yields relative to longer maturities, and corresponds to a tighter stance of monetary policy. Expected inflation also decreases with a higher expected unemployment rate but increases with a surprise increase in the unemployment rate. The VAR must assign the same weight and sign to the innovation and the expectation terms.

Inspecting the estimate for $\Sigma_{\mathcal{E}_{\mathcal{X}}}\Sigma'_{\mathcal{E}_{\mathcal{X}}}$ reveals the correlation between the updates of different conditional means. In the USA, the update of expected inflation is positively correlated with an update in expected unemployment and with an update in the slope factor. It is also negatively correlated with an expected increase in the level of yields. This effect is not statistically

[19] The autoregressive matrix $\hat{K}_{1X} + I_{\mathcal{N}_Z}$ has no unit root in both cases.

Table VI. Parameter estimates—US data

Parameter estimates for the no-arbitrage CM—DTSM model. Panel A reports parameters for the historical dynamics, $K_{1X} + I_{\mathcal{N}_Z}$ and $\Sigma_{\mathcal{E}_X}\Sigma_X^{-1}$, respectively. Panel B reports the correlation of innovations to X_t and innovations to $\mathcal{E}_{\mathcal{X},t}$ implicit in the matrices $\Sigma_X\Sigma_X'$ and $\Sigma_{\mathcal{E}_X}\Sigma_{\mathcal{E}_X}'$, respectively. Diagonal elements correspond to the standard deviations in annualized basis points. Panel C reports parameters for the risk-neutral dynamics, $K_{1X}^{Q} + I_{\mathcal{N}_Z}$. Asymptotic standard errors are in parentheses when applicable. Monthly data, December 1985 to December 2012.

Panel A: historical dynamics

$$K_{1X} + I_{\mathcal{N}_Z} = \begin{pmatrix} 0.98 & -0.04 & 0.01 & -0.02 \\ (0.03) & (0.02) & (0.003) & (0.02) \\ 0.20 & 0.84 & -0.01 & -0.06 \\ (0.10) & 0.05) & (0.01) & (0.3) \\ -1.49 & 0.64 & 1.08 & 0.17 \\ (0.36) & (0.16) & (0.4) & (0.09) \\ -0.01 & -0.003 & 0.02 & 0.97 \\ (0.09) & (0.04) & (0.01) & (0.02) \end{pmatrix} \qquad \Sigma_{\mathcal{E}_X}\Sigma_X^{-1} = \begin{pmatrix} -0.01 & 0.51 & 0.01 & 0.02 \\ (0.001) & (0.06) & (0.01) & (0.04) \\ -0.02 & 0.95 & -0.10 & -0.31 \\ (0.004) & (0.05) & (0.01) & (0.04) \\ 0.001 & -0.19 & 1.01 & -0.08 \\ (0.002) & (0.06) & (0.4) & (0.03) \\ 0.0003 & -0.01 & 0.02 & 0.98 \\ (0.001) & (0.02) & (0.004) & (0.01) \end{pmatrix}$$

Panel B: standard deviations and correlations

$$\Sigma_X\Sigma_X' \rightarrow \begin{pmatrix} 3.23 & 0.01 & 0.14 & -0.15 \\ (0.07) & (0.07) & (0.10) & (0.07) \\ & 0.26 & -0.13 & 0.21 \\ & (0.01) & (0.07) & (0.07) \\ & & 1.63 & -0.70 \\ & & (0.05) & (0.02) \\ & & & 0.45 \\ & & & (0.01) \end{pmatrix} \qquad \Sigma_{\mathcal{E}_X}\Sigma_{\mathcal{E}_X}' \rightarrow \begin{pmatrix} 0.13 & 0.87 & -0.16 & 0.25 \\ (0.02) & (0.04) & (0.18) & (0.16) \\ & 0.28 & -0.43 & 0.13 \\ & (0.02) & (0.07) & (0.08) \\ & & 1.68 & -0.69 \\ & & (0.06) & (0.02) \\ & & & 0.42 \\ & & & (0.02) \end{pmatrix}$$

Panel C: risk-neutral dynamics

$$K_{1X}^{Q} + I_{\mathcal{N}_Z} = \begin{pmatrix} 0.83 & 0.05 & 0.02 & 0.003 \\ (0.02) & (0.01) & (0.002) & (0.01) \\ -0.09 & 1.02 & 0.01 & 0.01 \\ (0.04) & (0.01) & (0.004) & (0.01) \\ -0.01 & 0.0001 & 1.00 & -0.07 \\ (0.03) & (0.01) & (0.003) & (0.003) \\ -0.22 & 0.05 & 0.03 & 0.98 \\ (0.04) & (0.02) & (0.004) & (0.01) \end{pmatrix}$$

significant. Therefore, a standard expectation update sees higher expected inflation, a flatter expected curve (the slope factor pushes short-maturity yields up), and higher expected unemployment. In contrast, inflation innovations are uncorrelated with unexpected unemployment changes.

In Canada, a standard expectation update also sees higher expected inflation, a flatter expected curve, and higher expected unemployment. However, the correlation with the expected level of yield is stronger in Canada. In addition, inflation innovations exhibit a low correlation with

Table VII. Parameter estimates—Canadian data

Parameter estimates for the no-arbitrage CM—DTSM model. Panel A reports parameters for the historical dynamics, $K_{1X}+I_{\mathcal{N}_Z}$ and $\Sigma_{\mathcal{E}_X}\Sigma_X^{-1}$, respectively. Panel B reports the correlation of innovations to X_t and innovations to $\mathcal{E}_{\mathcal{X},t}$ implicit in the matrices $\Sigma_X\Sigma_X'$ and $\Sigma_{\mathcal{E}_X}\Sigma_{\mathcal{E}_X}'$, respectively. Diagonal elements correspond to the standard deviations in annualized basis points. Panel C reports parameters for the risk-neutral dynamics, $K_{1X}^Q+I_{\mathcal{N}_Z}$. Asymptotic standard errors are in parentheses when applicable. Macro and yield data 01/1992 to 12/2012. Survey data Q1-1992/Q4-2012.

Panel A: historical dynamics

$$K_{1X}+I_{\mathcal{N}_Z}=\begin{pmatrix} 1.03 & -0.07 & 0.003 & -0.04 \\ (0.15) & (0.06) & (0.003) & (0.04) \\ 0.36 & 0.82 & 0.003 & -0.10 \\ (0.25) & (0.11) & (0.005) & (0.07) \\ -1.28 & 0.68 & 0.969 & 0.36 \\ (1.16) & (0.51) & (0.026) & (0.31) \\ -0.30 & 0.02 & 0.000 & 0.99 \\ (0.21) & (0.10) & (0.005) & (0.06) \end{pmatrix} \qquad \Sigma_{\mathcal{E}_X}\Sigma_X^{-1}=\begin{pmatrix} -0.005 & 0.35 & -0.03 & 0.26 \\ (0.002) & (0.07) & (0.01) & (0.06) \\ -0.011 & 0.73 & -0.04 & -0.04 \\ (0.002) & (0.06) & (0.01) & (0.03) \\ 0.000 & -0.01 & 0.99 & -0.05 \\ (0.002) & (0.08) & (0.02) & (0.09) \\ 0.002 & -0.10 & 0.01 & 0.90 \\ (0.001) & (0.03) & (0.01) & (0.03) \end{pmatrix}$$

Panel B: standard deviations and correlations

$$\Sigma_X\Sigma_X' \rightarrow \begin{pmatrix} 4.01 & 0.02 & 0.08 & 0.01 \\ (0.16) & (0.08) & (0.08) & (0.08) \\ & 0.18 & -0.07 & 0.04 \\ & (0.01) & (0.07) & (0.07) \\ & & 1.09 & -0.04 \\ & & (0.05) & (0.06) \\ & & & 0.44 \\ & & & (0.02) \end{pmatrix} \qquad \Sigma_{\mathcal{E}_X}\Sigma_{\mathcal{E}_X}' \rightarrow \begin{pmatrix} 0.14 & 0.47 & -0.31 & 0.80 \\ (0.03) & (0.10) & (0.10) & (0.06) \\ & 0.15 & -0.34 & -0.14 \\ & (0.01) & (0.08) & (0.11) \\ & & 1.08 & -0.03 \\ & & (0.12) & (0.07) \\ & & & 0.39 \\ & & & (0.03) \end{pmatrix}$$

Panel C: risk-neutral dynamics

$$K_{1X}^Q+I_{\mathcal{N}_Z}=\begin{pmatrix} 0.78 & 0.08 & -0.001 & 0.03 \\ (0.14) & (0.06) & (0.002) & (0.04) \\ 0.24 & 0.89 & -0.001 & -0.10 \\ (0.40) & (0.17) & (0.004) & (0.11) \\ -0.60 & 0.21 & 0.99 & 0.02 \\ (0.30) & (0.13) & (0.01) & (0.08) \\ -0.68 & 0.26 & -0.003 & 1.12 \\ (0.39) & (0.17) & (0.007) & (0.10) \end{pmatrix}$$

other innovations: monthly inflation appears largely unpredictable in Canada. Again, the VAR cannot capture these separate effects.

4.5 NOMINAL YIELD DECOMPOSITIONS

4.5.a. *Fisher's decomposition*

Real yields are identified within the model if the expected inflation and the inflation-risk premium can be derived from the model. Hence, if the inflation rate is included in the $\mathbb{P}$-dynamics, and if the inflation-risk premium is identified, it follows that real yields are identified. The only additional assumption is the absence of an arbitrage opportunity in the real bond market. The real short rate r_t follows from the link between the nominal and the real stochastic discount factors, SDF_{t+1} and SDF^r_{t+1}, respectively. This link, $\mathrm{SDF}_{t+1} \equiv e^{-i_t}\xi_{t+1} = e^{-\pi_{t+1}}\mathrm{SDF}^r_{t+1}$, implies that

$$r_t \equiv -\log\left(E_t[\mathrm{SDF}^r_{t+1}]\right) = i_t - E^Q_t[\pi_{t+1}] - \frac{1}{2}\sigma^2_\pi, \tag{47}$$

$$r_t = \rho^r_{0X} + \rho^{r'}_{1X}\mathcal{E}_{\mathcal{X},t}, \tag{48}$$

and it follows that real yields are given by

$$r^{(n)}_t = a_{r,n} + b'_{r,n}\mathcal{E}_{\mathcal{X},t}, \tag{49}$$

with the coefficients ρ^r_{0X} and ρ^r_{1X} and the loadings $a_{r,n}$ and $b_{r,n}$ given in Appendix A1. For longer-maturity yields, we have the following generalized Fisher decomposition:

$$i^{(n)}_t \equiv c^{(n)} + r^{(n)}_t + \mathcal{E}^{(n)}_{\pi,t} + irp^{(n)}_t, \tag{50}$$

where $r^{(n)}_t$ is the n-period real yield, $irp^{(n)}_t$ is the inflation-risk premium, $\mathcal{E}^{(n)}_{\pi,t}$ is the inflation expectation, and $c^{(n)}$ is a Jensen term (see Appendix A5).

4.5.b. *Inflation swaps, risk premium, and Sharpe ratio*

Real yields may be imprecisely estimated even if formally identified, and using inflation swap rates helps pin down the decomposition in the USA. However, the additional data are not reliably available in most countries (Canada is a typical case). To improve estimates of the real yield in these cases, we constrain the inflation Sharpe ratio within a plausible area of the parameter space (see Section 3.2.6).

We ask if the inflation-risk premium estimates are similar whether we use inflation swaps at estimation or exclude them. To check this, we reestimate the CM model with US data, excluding the inflation swaps from the joint likelihood (Case B), but using the same constraint on the conditional Sharpe ratio as in the case of Canada. We focus on the subsample where US markets were relatively unaffected by liquidity shocks and policy intervention. The financial crisis and the rounds of quantitative easing introduce a wedge between the risk premium implicit in Treasury yields and the risk premium implicit in the inflation swaps (Krishnamurthy and Vissing-Jorgensen, 2011; Fleckenstein, Longstaff, and Lustig, 2013). Estimates based on post-2007 will reflect that wedge. Therefore, we proceed with data between 1985 and 2007.

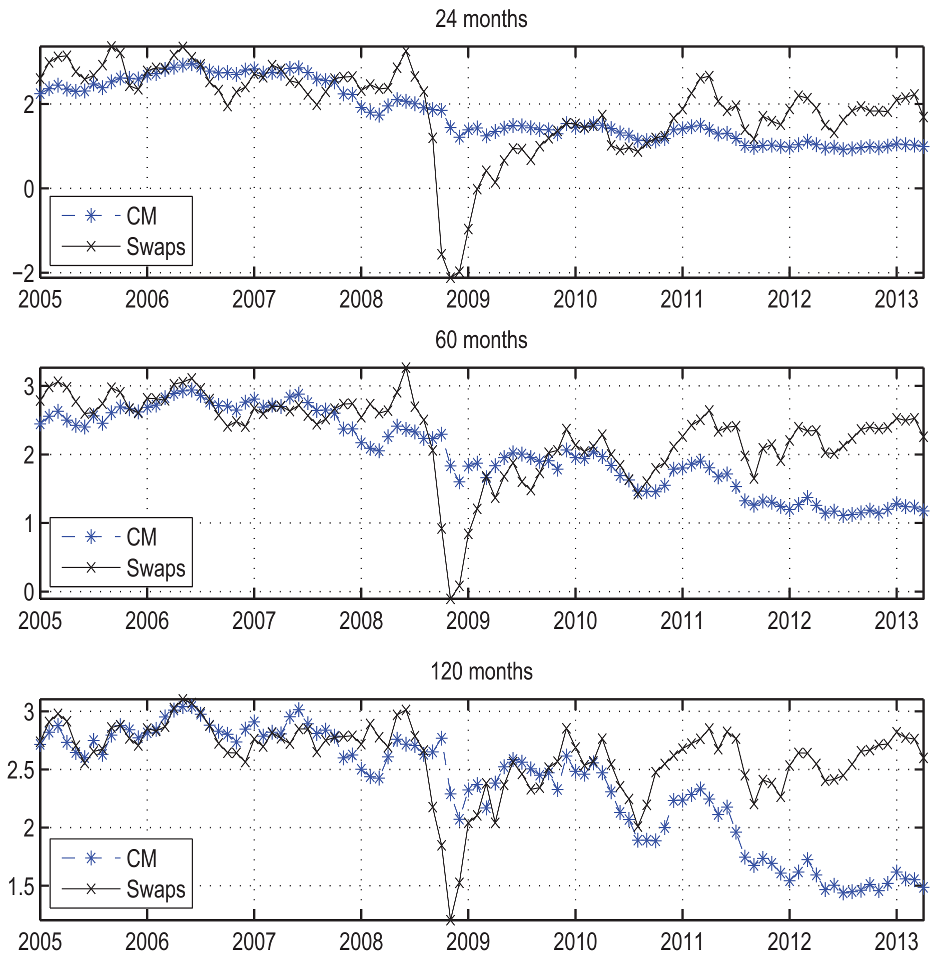


Figure 4. Inflation swap rates**.** Observed and model-implied inflation swap rates between 2005 and 2013. The model-implied values are computed from the benchmark CM model estimated using data until the end of 2007, excluding inflation swap data altogether (Case B), and bounding the distribution of the conditional inflation Sharpe ratio.

Figure 4 compares the model-implied and observed inflation swap rates for 2-, 5-, and 10-year maturities, and extends the estimation sample until 2012. The swap rates predicted from the model are close to the observed rates. In fact, excluding the swap data seems to help in the crisis. The model-implied rates do not plunge like observed rates do (as low as −2% in the case of the 2-year swap). This must be assessed as a benefit of using the Sharpe ratio constraint. The model-implied and observed rates then remain close to each other until the end of 2010, which coincides with the second round of

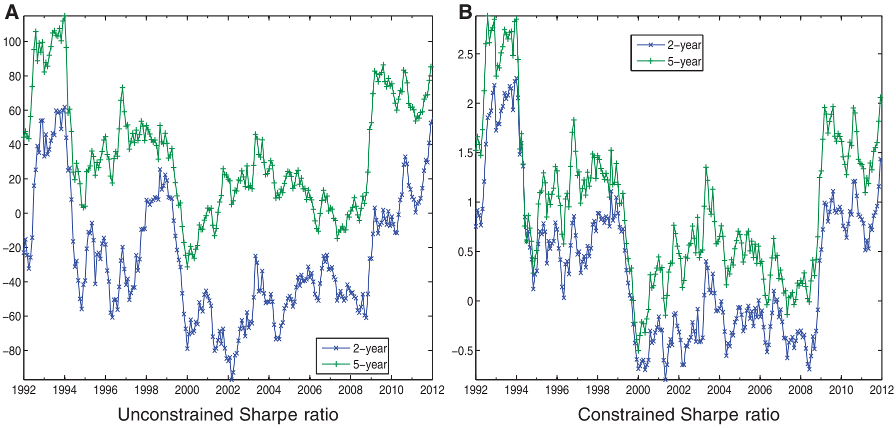


Figure 5. Inflation-risk premiums and the Sharpe ratio constraint. The 2-year and 5-year inflation-risk premium in Canada from the CM model where the Sharpe ratio is unconstrained (A) and where it is constrained with $\kappa = 0.2$ (B).

quantitative easing. At this point, the model-implied rates follow nominal yields, slowly drifting downward, while inflation swap rates remain high. The wedge between the model and the swap markets is similar across maturities (note the different scales). Including the swap data is beneficial in this case. Overall, the results suggest that bounding the inflation Sharpe ratio is a close substitute to using the inflation swap data.

Figure 5 illustrates the impact of the Sharpe ratio constraint on the yield decomposition in Canada. Figure 5A shows the 2- and 5-year inflation-risk premium obtained when the Sharpe ratio is unconstrained. Figure 5B shows the results with the constraint set to $\kappa = 0.2$.[20] The excess variability of the inflation-risk premium stands out in the unconstrained case: the 2-year inflation-risk premium varies between −20% and 5%, and the 5-year inflation-risk premium varies between −5% and 15%! These large values imply implausibly large variations in the real yields. In contrast, the constrained model produces estimates of the inflation-risk premium that are economically plausible and close to the US estimates, ranging between 0% and 2%.

[20] We assess the variability of several other model properties as we vary κ. The short-rate loadings, yield loadings, measurement errors, and forecasting performance remain essentially unchanged for values of κ around 0.2. However, the likelihood increases monotonically as we increase κ to implausible values, and with very little improvement in pricing errors—a telling tale of overfitting. The risk premium variability also increases monotonically. Results are available from the authors.

4.5.c. *Fisher's decomposition in CM and VAR models*

This section assesses how the decomposition of US nominal yields changes as the information set used for estimation changes. We report results for the 2-year yield from the CM and the VAR models across Cases A and B. The yield decompositions are very similar across estimates of the CM model, but the estimates vary across VAR models.

The left column of Figure 6 reports results for the CM model. The path of 2-year expected inflation is very similar across cases. The estimated inflation-risk premium becomes more volatile when we add inflation swap data. This reflects the fact that we do not enforce the constraint on the Sharpe ratio in this case. Also, the inflation-risk premium rises in 2011 and 2013. As noted in Section 4.5.2, unconventional policies introduce a wedge between nominal yields and inflation swaps. Otherwise, the results are remarkably similar.

The right column of Figure 6 reports results for the VAR model. The message could not be more different. The estimates of the inflation-risk premium become erratic when we exclude the inflation swap. Even when we include all the data, in Case A, the estimates of the inflation-risk premium differ from the CM model, reflecting the poor fit of the swap data documented in Table IIIA. Results from the VAR are not reliable.

4.5.d. *Nominal yield decomposition in the USA and Canada*

Figure 7 compares the decompositions from our benchmark model in the USA and in Canada.[21] It displays the 2-year nominal yield, expected inflation, and inflation-risk premium. Figure 7A reports the decomposition in the USA, reproducing the results from Figure 6A. Figure 7B reports the decomposition the Canada. For the reasons discussed in Section 4.5.2, we use parameters estimated with data until 2007 for the USA.

Looking beyond the cyclical variations, we see that the inflation-risk premium in both countries had been drifting down until 2008, from around 1% to around 0% in the USA and from 2% to 0% in Canada. The inflation-risk premium is procyclical in the USA, rising when nominal

[21] These are the first real yield estimates for Canada. Ragan (1995) is an early attempt to measure expected inflation from nominal yields in Canada. Day and Lange (1997) provide time-series evidence that the term spread has predictive content for inflation rates in Canada. Fung, Mittnick, and Remolona (1999) study a joint model for the USA and Canadian term structures. They do not use survey data to identify inflation expectations. Garcia and Luger (2007) estimate an equilibrium-based model for yields in Canadian data where conditional expectations play a central role.

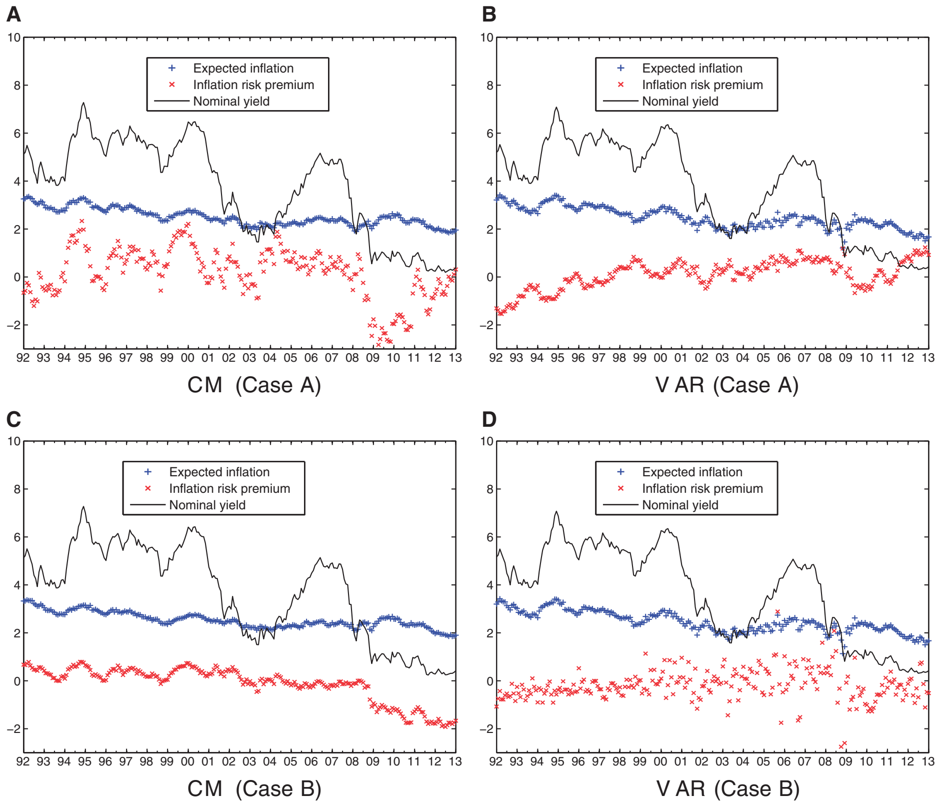


Figure 6. Nominal yield decomposition. Decomposition of the 2-year nominal yields between January 1992 and December 2012 in the USA from different estimates of the CM model (left column) and the VAR model (right column). (A) and (B) report results for Case A estimated using yield, inflation swap, and survey inflation forecast data. (D) and (C) report results for Case B, estimated excluding the swap data.

yields rise, and dampening the transmission of interest rate shocks to the real curve. Real yields are less cyclical than nominal yields in the USA. This is consistent with results in, for example, Chernov and Mueller (2011).[22] In contrast, the inflation-risk premium is countercyclical in Canada, amplifying the transmission of interest rate shocks to the real curve. Real yields are more cyclical than nominal yields in Canada. This stark difference between the two countries may arise because of the different mandates given to their respective central banks. The Bank of Canada has had an explicit inflation

22 See the results for “AO5” models in their Figure 4.

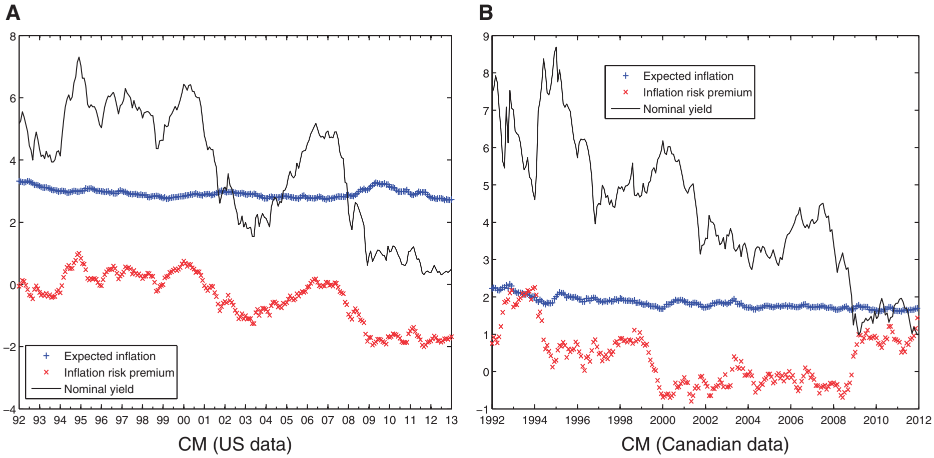


Figure. 7. Nominal yield decomposition. Decomposition of the 2-year nominal yields between January 1992 and December 2012 in the USA (A) and in Canada (B). Results from the CM model estimated with yields, inflation swaps, and inflation survey forecasts (Case A).

target since the beginning of our sample, but the Fed has had a dual mandate, weighting employment and inflation targets. We leave the causal mechanism underlying this difference to future research, but the results suggest that monetary policy achieves a greater bang for the buck in Canada.

This difference implies a very different response of the inflation-risk premium to the large declines of long-term yields in both countries since 2009. In the USA, the model-implied premium followed the decline in yields during the crisis and has remained between −1.5% and −2.0% since then. Exposures to inflation risk are perceived as hedges by US bond investors. In contrast, the inflation-risk premium increased to 1% between 2009 and 2010 in Canada and has remained elevated since. Exposures to inflation shocks are perceived as risky by Canadian bond investors. Again, the difference may be linked to the difference between the Fed's and the Bank of Canada's mandate. The Fed may be perceived to accommodate inflation shocks to fulfill its dual mandate, as least in the early stage of an expansion, while the Bank of Canada may be perceived to effectively counter inflation shocks. This interpretation is consistent with the lower volatility of expected inflation between the USA and Canada in Figure 2. David and Veronesi (2013) provide empirical evidence that inflation shocks can be good news and bad news in the context of a general equilibrium. This effect may also arise with Epstein–Zin preferences: inflation may signal higher consumption

Table VIII. Model-implied summary statistics

Sample summary statistics in Canadian data for the nominal yield $y_t^{(n)}$, the real yield $r_t^{(n)}$, the expected inflation $\mathcal{E}_{\pi,t}^{(n)}$, and the inflation-risk premium, $\text{irp}_{t,n}$, at different maturities n, in months, computed from the model. The Sharpe ratio constraint is set to $\kappa = 0.2$.

Panel A: 3 months					Panel B: 6 months				
	$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$		$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$
Mean	4.89	3.16	1.92	−0.20	Mean	4.98	3.19	1.93	−0.14
Std. Dev.	3.32	3.43	0.37	0.70	Std. Dev.	3.22	3.35	0.33	0.73
$\rho(1)$	0.99	0.99	0.95	0.98	$\rho(1)$	0.99	0.99	0.95	0.98
$\rho(12)$	0.81	0.79	0.58	0.65	$\rho(12)$	0.82	0.79	0.62	0.66
Panel C: 1 year					**Panel D: 2 years**				
	$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$		$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$
Mean	5.13	3.22	1.93	−0.02	Mean	5.36	3.22	1.92	0.22
Std. Dev.	3.06	3.23	0.29	0.78	Std. Dev.	2.87	3.10	0.24	0.84
$\rho(1)$	0.99	0.98	0.97	0.98	$\rho(1)$	0.98	0.98	0.98	0.97
$\rho(12)$	0.83	0.78	0.71	0.65	$\rho(12)$	0.84	0.77	0.82	0.62
Panel E: 5 years					**Panel F: 10 years**				
	$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$		$y_t^{(n)}$	$r_t^{(n)}$	$\mathcal{E}_{\pi,t}^{(n)}$	$\text{irp}_t^{(n)}$
Mean	5.77	3.01	1.88	0.81	Mean	6.20	2.53	1.85	1.54
Std. Dev.	2.63	2.91	0.17	0.87	Std. Dev.	2.47	2.75	0.12	0.82
$\rho(1)$	0.98	0.98	0.99	0.96	$\rho(1)$	0.99	0.98	0.99	0.96
$\rho(12)$	0.85	0.76	0.89	0.56	$\rho(12)$	0.86	0.77	0.89	0.53

growth, leading to upward revisions for utility value (through the continuation value).

Table VIII reports sample summary statistics for each component of nominal yield in Canada, and for maturities of 3 and 6 months, as well as 1, 2, 5, and 10 years. We focus on the case of Canada since summary statistics for the US nominal yields have been documented elsewhere. The average nominal yield curve is upward sloping—the slope between 3-month and 10-year yields averages close to 1.3% in our sample—but the curve exhibits significant variability throughout the cycle: the average volatility ranges from 3.28% to 2.44%. Real yields are downward sloping—the average slope is −0.7%—suggesting that investors perceive real bonds as hedges in Canada. The average expected inflation is flat, above 2%. Therefore, the average positive slope of nominal yields is due to the slope of the inflation-risk premium. The inflation-risk premium average ranges from almost zero at the shortest maturity to 1.6% at the 10-year maturity.

5. Conclusion

We introduce the class of CM-MTSMs where the conditional mean $\mathcal{E}_t$ drives the term structure of yields $y_t^{(n)} = a_n + b_n' \mathcal{E}_t$ as well as the dynamics of the state variables. The essence of our approach is the departure from the Markov assumption: $\mathcal{E}_t$ depend on the entire history of $\mathcal{Z}_t$. This extension is parsimonious and gives fundamentally different roles to $\mathcal{Z}_t$, which captures the current state, and to $\mathcal{E}_t$, which summarizes yields as well as the conditional mean of $\mathcal{Z}_{t+1}$. Including inflation within a CM-MTSM as a showcase, the results illustrate the distinctive properties of CM-MTSMs: (i) yields span expected inflation as measured by survey forecasts, and (ii) expected changes and surprises in unemployment and in the level or the slope have opposite effects on inflation expectations. A simpler Markovian VAR(1) model does not capture these features. In addition, our approach reconciles models using survey forecasts as state variables with models using the underlying macro variables as state variables. Importantly, including survey data overcomes the effect of sampling variability on the estimates of persistence parameters, and including inflation swap data helps us obtain plausible estimates of the inflation-risk premium, but estimates of the CM remain robust when the additional data are not available for estimation.

The dynamics in CM-MTSMs are consistent with several general equilibrium specifications where the observable macro variables are not Markovian (e.g., Bansal and Yaron, 2004; Bansal and Shaliastovich, 2013). Our results suggest that (non-Markovian) long-run risk models with a structural pricing kernel calibrated to reasonable Sharpe ratio may fit term structure data (e.g., Bansal and Shaliastovich, 2013). In that context, our results raise the question as to why inflation shocks may be perceived as a hedge in the USA since 2009, but perceived as a risk in Canada. We also leave important extensions for future research. First, we focus on relatively short horizons, of less than 10 years. Kozicki and Tinsley (2001, 2006) highlight the role of shifting endpoints in the evolution of long-horizon inflation rates in US data. In Canada, Amano and Murchison (2006) show the importance of a shifting endpoint for inflation in the early 1990s during the transition toward a 2% inflation-targeting regime. Second, we focus on inflation forecasts, but forecasts of future inflation rates are not independent of forecasts of other macro forecasts. Extending our approach to include additional variables, such as unemployment, could help identify the dynamic relationships between economic activity, inflation, and risk as perceived by bond investors. The framework proposed here can be extended in these directions.

Appendix A

A1 YIELD COEFFICIENT RECURSIONS

The price of a nominal zero-coupon bond with maturity n is given by $D(t,n)=\exp\left(A_n+B_n'\mathcal{E}_t\right)$ with coefficients given by

$$\begin{aligned} B_{n+1} &= (I_{\mathcal{N}_Z}+K_1^{\mathbb{Q}'})B_n-\rho_1 \\ A_{n+1} &= A_n+K_0^{\mathbb{Q}'}B_n+\frac{1}{2}B_n'\Sigma_{\mathcal{E}}\Sigma_{\mathcal{E}}'B_n-\rho_0, \end{aligned} \tag{A.1}$$

for $n>0$ and with initial conditions $A_0=0$ and $B_0=0$. The nominal yield coefficients are given by $a_n\equiv -A_n/n$ and $b_n\equiv -B_n/n$. From Equations (24) and (47), it follows that the real rate coefficients in Equation (48) are given by

$$\begin{aligned} \rho_{0\mathcal{X}}^r &\equiv \rho_{0\mathcal{X}}-\frac{1}{2}\sigma_\pi^2-e_1'\Sigma_X\Sigma_{\mathcal{E}_{\mathcal{X}}}^{-1}\left(K_{0\mathcal{X}}^{\mathbb{Q}}-K_{0X}^{\mathbb{P}}\right) \\ \rho_{1\mathcal{X}}^r &\equiv \rho_{1\mathcal{X}}-e_1-\left[\Sigma_X\Sigma_{\mathcal{E}_{\mathcal{X}}}^{-1}\left(K_{1X}^{\mathbb{Q}}-K_{1X}^{\mathbb{P}}\right)\right]'e_1. \end{aligned} \tag{A.2}$$

A2 RISK PREMIUM

From Equations (8), (13), and (12), it follows that

$$\begin{aligned} \mathcal{E}_t^{\mathbb{Q}} &\equiv E_t^{\mathbb{Q}}[\mathcal{Z}_{t+1}] \\ &= \mathcal{E}_t+\Sigma E_t^{\mathbb{Q}}[\epsilon_{t+1}^{\mathbb{P}}] \\ &= \mathcal{E}_t+\Sigma\Sigma_{\mathcal{E}}^{-1}E_t^{\mathbb{Q}}[\Delta\mathcal{E}_{t+1}-K_0^{\mathbb{P}}-K_1^{\mathbb{P}}\mathcal{E}_t] \\ &= \mathcal{E}_t+\Sigma\Sigma_{\mathcal{E}}^{-1}\left(K_0^{\mathbb{Q}}+K_1^{\mathbb{Q}}\mathcal{E}_t-K_0^{\mathbb{P}}-K_1^{\mathbb{P}}\mathcal{E}_t\right) \\ &= \mathcal{E}_t+\Sigma\Sigma_{\mathcal{E}}^{-1}\left(K_0^{\mathbb{Q}}-K_0^{\mathbb{P}}\right)+\Sigma\Sigma_{\mathcal{E}}^{-1}\left(K_1^{\mathbb{Q}}-K_1^{\mathbb{P}}\right)\mathcal{E}_t^{\mathbb{P}}. \end{aligned} \tag{A.3}$$

A3 PROPOSITION 2

The $\mathcal{N}_Z\times 1$ vector of observable $\mathcal{X}_t$ defined in Equations (19)–(20) has the same dynamics as the process for $\mathcal{Z}_t$. We define D:

$$D\equiv D_1+D_2\left(K_1^{\mathbb{P}}+I_{\mathcal{N}_{\mathcal{Z}}}\right), \tag{A.4}$$

and we require that D be invertible. Using the mapping between $\mathcal{X}_t$ and $\mathcal{Z}_t$ given in Equation (19), the parameters for the dynamics of $\mathcal{X}_t$ are given by

$$\begin{aligned} K_{0X}^{\mathbb{P}} &= DK_0^{\mathbb{P}}-DK_1^{\mathbb{P}}D^{-1}\left(C+D_2K_0^{\mathbb{P}}\right) \\ K_{1X}^{\mathbb{P}} &= DK_1^{\mathbb{P}}D^{-1}. \end{aligned} \tag{A.5}$$

$$K_{0X}^{\mathbb{Q}} = DK_0^{\mathbb{Q}} - DK_1^{\mathbb{Q}} D^{-1}\left(C + D_2 K_0^{\mathbb{P}}\right)$$
$$K_{1X}^{\mathbb{Q}} = DK_1^{\mathbb{Q}} D^{-1}. \tag{A.6}$$
$$\Sigma_{\mathcal{E}_{\mathcal{X}}} = D\Sigma_{\mathcal{E}}$$
$$\Sigma_X = D_1 \Sigma + D_2 \Sigma_{\mathcal{E}}. \tag{A.7}$$

A4 THEOREM 1

The canonical form for $\mathcal{Z}_t$ in Proposition 1 is exactly identified. We verify that the parameterization for $\mathcal{X}_t$ in Theorem 1 is also identified. First, we redefine D in Equation (A.4) to substitute out the matrix $K_1^{\mathbb{P}}$, which belongs to the dynamics of $\mathcal{Z}_t$. Given $D \equiv D_1 + D_2\left(K_1^{\mathbb{P}} + I_{\mathcal{N}_{\mathcal{Z}}}\right)$ and using $K_{1X}^{\mathbb{P}} = DK_1^{\mathbb{P}} D^{-1}$ from Proposition 2, it follows that

$$\begin{aligned} D &= D_1 + D_2\left(K_1^{\mathbb{P}} + I_{\mathcal{N}_{\mathcal{Z}}}\right) \\ &= D_1 + D_2 + D_2 D^{-1} K_{1X}^{\mathbb{P}} D, \end{aligned} \tag{A.8}$$

where D_1 and D_2 are defined in Equation (20). Rearranging:

$$I_N = (D_1 + D_2)D^{-1} + D_2 D^{-1} K_{1X}^{\mathbb{P}}$$
$$\mathrm{vec}(I_N) = \mathrm{vec}\left((D_1 + D_2)D^{-1}\right) + \mathrm{vec}\left(D_2 D^{-1} K_{1X}^{\mathbb{P}}\right)$$
$$\mathrm{vec}(I_N) = (I_N \otimes (D_1 + D_2))\mathrm{vec}\left(D^{-1}\right) + \left(K_{1X}^{\mathbb{P}\prime} \otimes D_2\right)\mathrm{vec}\left(D^{-1}\right), \tag{A.9}$$

and, therefore, D is defined implicitly by

$$\left(I_N \otimes D_1 + \left(I_N + K_{1X}^{\mathbb{P}\prime}\right) \otimes D_2\right)\mathrm{vec}\left(D^{-1}\right) = \mathrm{vec}(I_N).$$

We require that

$$\left(I_{\mathcal{N}_{\mathcal{Z}}} \otimes D_1 + \left(I_{\mathcal{N}_{\mathcal{Z}}} + K_{1X}^{\mathbb{P}\prime}\right) \otimes D_2\right) \tag{A.10}$$

be invertible so that D^{-1} is well-defined. Second, we explicitly provide the mappings between the parameters of $\mathcal{Z}_t$ and the parameters of $\mathcal{X}_t$. The mapping between the $\mathbb{P}$ parameters is given by

$$\begin{aligned} K_0^{\mathbb{P}} &= \left(D - K_{1X}^{\mathbb{P}} D_2\right)^{-1}\left(K_{0X}^{\mathbb{P}} + K_{1X}^{\mathbb{P}} C\right) \\ K_1^{\mathbb{P}} &= D^{-1} K_{1X}^{\mathbb{P}} D \\ \Sigma_{\mathcal{E}} &= D^{-1} \Sigma_{\mathcal{E}_{\mathcal{X}}} \\ D_1 \Sigma &= \Sigma_X - D_2 D^{-1} \Sigma_{\mathcal{E}_{\mathcal{X}}}, \end{aligned} \tag{A.11}$$

and the mapping for the $\mathbb{Q}$ parameters is given by

$$K_{0X}^{\mathbb{Q}} = DK_0^{\mathbb{Q}} - DJ(\lambda^{\mathbb{Q}})(D_1 + D_2)^{-1}(C + D_2 D^{-1} K_{0X}^{\mathbb{P}})$$
$$K_{1X}^{\mathbb{Q}} = DJ(\lambda^{\mathbb{Q}})D^{-1}. \tag{A.12}$$

See Proposition 1. Importantly, both sides of the last equation in (A.11) have the same rank as D_1, which cannot be inverted since its lower block is zero. In other words, we cannot identify Σ_X and $\Sigma_{\mathcal{E}_\mathcal{X}}$ separately whenever yield portfolios are included among the observable variables. To complete the parameterization in the theorem, rewrite

$$D_1 \Sigma = \begin{bmatrix} \Sigma^* \\ 0 \end{bmatrix}, \tag{A.13}$$

defining Σ^* implicitly. Then, Σ_X is given by the last equation in (A.11) in terms of the other parameters,

$$\Sigma_X = D_2 D^{-1} \Sigma_{\mathcal{E}_\mathcal{X}} + \begin{bmatrix} \Sigma^* \\ 0 \end{bmatrix}. \tag{A.14}$$

Turning to the short rate, from Equations (19) to (20) and Proposition 1, it follows that the coefficients in Equation (21) are given by

$$\rho_{0\mathcal{X}} = -\imath'(C + D_2 K_0^{\mathbb{P}})$$
$$\rho_{1\mathcal{X}}' = \imath' D^{-1}. \tag{A.15}$$

The dynamics of $\mathcal{X}_t$ in Equations (22)–(23) follow from Proposition 2. Finally, the spread between $\mathcal{E}_{\mathcal{X},t}^{\mathbb{Q}}$ and $\mathcal{E}_{\mathcal{X},t}^{\mathbb{P}}$ in Equation (24) follows from Proposition 2 and Equation (17).

A5 GENERALIZED FISHER DECOMPOSITION

The generalized Fisher decomposition can be applied to any nominal yield with maturity n,

$$i_t^{(n)} \equiv c^{(n)} + r_t^{(n)} + \mathcal{E}_t^{\pi,(n)} + \mathrm{irp}_t^{(n)}, \tag{A.16}$$

where $r_t^{(n)}$ is the real yield, and $\mathrm{irp}_t^{(n)}$ is the n-period-ahead inflation-risk premium,

$$\mathrm{irp}_t^{(n)} \equiv \frac{1}{n}\left(E_t^Q\left[\sum_{j=1}^{n} \pi_{t+j}\right] - E_t\left[\sum_{j=1}^{n} \pi_{t+j}\right]\right),$$

where $\mathcal{E}_t^{\pi,(n)}$ is the n-period-ahead inflation expectation,

$$\mathcal{E}_t^{\pi,(n)} \equiv E_t^{\mathbb{P}}\left[\frac{\sum_{j=1}^{n}\pi_{t+j}}{n}\right],$$

and where $c^{(n)}$ is a Jensen term,

$$c^{(n)} = \frac{1}{2}\sigma_\pi^2 + \frac{1}{2n}\sum_{j=0}^{n-1} j^2\left(b'_{r,j}\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma'_{\mathcal{E}_\mathcal{X}}b_{r,j} - b'_j\Sigma_{\mathcal{E}_\mathcal{X}}\Sigma'_{\mathcal{E}_\mathcal{X}}b_j\right).$$

A6 COMBINING INFLATION AND SURVEY INFLATION FORECASTS

Rearrange the vector $\mathcal{M}_t$ of macroeconomic variables as $\mathcal{M}_t = [\mathcal{O}'_t\ \mathcal{S}'_t]'$ where $\mathcal{O}_t$ stacks the macro data (i.e., the inflation rate) and $\mathcal{S}_t$ stacks the corresponding survey data (i.e., the inflation survey). As before, the observed variables are linked to the latent factors as in Equation (18),

$$\begin{aligned}\mathcal{O}_t &= \gamma_{0\mathcal{O}} + \gamma'_{\mathcal{O}}\mathcal{Z}_t\\ \mathcal{S}_t &= \gamma_{0\mathcal{S}} + \gamma'_{\mathcal{S}}\mathcal{E}_t,\end{aligned} \tag{A.17}$$

but changing the notation slightly. We derive parametric restrictions such that we have

$$\mathcal{S}_t = E_t^{\mathbb{P}}[\mathcal{O}_{t+1}] + \eta_t, \tag{A.18}$$

where η_t is a measurement error: $E[\eta_t] = 0$ and $\text{cov}(\eta_t, \eta_{t+h}) = 0 \quad \forall h > 0$. This implies restriction on the $\mathbb{P}$-dynamics of $\mathcal{X}_t$. Define $Y'_t = [\mathcal{X}'_t\ \mathcal{E}'_{\mathcal{X},t}]$ with dynamics given by

$$Y_{t+1} = \mu_Y + \phi_Y Y_t + \epsilon_{Y,t}, \tag{A.19}$$

and parameters given by

$$\mu_y = \begin{bmatrix} 0 \\ K_{0X}^{\mathbb{P}} \end{bmatrix} \quad \phi_Y = \begin{bmatrix} 0 & I \\ 0 & (I + K_{1X}^{\mathbb{P}}) \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & \phi_X \end{bmatrix}. \tag{A.20}$$

Define $e'_{\mathcal{O}}$ and $e'_{\mathcal{S}}$ such that $e'_{\mathcal{O}}\mathcal{X}_t = \mathcal{O}_t$ and $e'_{\mathcal{S}}\mathcal{X}_t = \mathcal{S}_t$. Then, $\eta_t = e'_{\mathcal{S}}\mathcal{X}_t - e'_{\mathcal{O}}\mathcal{E}_{\mathcal{X},t} = \beta' Y_t$ with $\beta' \equiv [e'_{\mathcal{S}}\ \ -e'_{\mathcal{O}}]$ and, therefore, $\text{cov}(\eta_t, \eta_{t+h})$ is given by

$$\begin{aligned}\text{cov}(\beta' Y_t, \beta' Y_{t+h}) &= \beta' \text{cov}(Y_t, Y_{t+h})\beta = \beta'\Omega_Y(\phi'_Y)^h\beta\\ &= \beta'\Omega_Y\begin{bmatrix} 0 & 0 \\ (\phi'_X)^{h-1} & (\phi'_X)^{h-1}\end{bmatrix}\beta \quad \forall h > 1,\end{aligned} \tag{A.21}$$

where we define the following implicitly:

$$\Omega_Y \equiv \begin{bmatrix} \mathrm{var}(\mathcal{X}_t) & \mathrm{cov}(\mathcal{X}_t, \mathcal{E}_{\mathcal{X},t}) \\ \mathrm{cov}(\mathcal{E}_{\mathcal{X},t}, \mathcal{X}_t) & \mathrm{var}(\mathcal{E}_{\mathcal{X},t}) \end{bmatrix} \equiv \begin{bmatrix} \Omega_X & \Omega_{X\mathcal{E}} \\ \Omega_{\mathcal{E}X} & \Omega_{\mathcal{E}} \end{bmatrix}. \tag{A.22}$$

Hence,

$$\mathrm{cov}(\eta_t, \eta_{t+h}) = \left(e'_{\mathcal{S}}\Omega_{X\mathcal{E}} - e'_{\mathcal{O}}\Omega_{\mathcal{E}}\right)(\phi'_X)^{h-1}\delta, \tag{A.23}$$

where we define

$$\delta \equiv e_{\mathcal{S}} - \phi'_X e_{\mathcal{O}}. \tag{A.24}$$

Clearly, $\delta = 0 \Rightarrow \mathrm{cov}(\eta_t, \eta_{t+h}) = 0 \quad \forall h > 0$, which is the condition that we discuss in the main text. Developing Equation (A.23), we can derive additional sufficient conditions. Tedious algebra yields

$$\mathrm{cov}(\eta_t, \eta_{t+h}) = \left(\delta\Omega_X\phi'_X + e'_{\mathcal{O}}\theta_X\Omega_\epsilon\phi'_X - (\delta + \theta'_X e_{\mathcal{O}})'\Omega_\epsilon\theta'_X\right)(\phi'_X)^{h-1}\delta, \tag{A.25}$$

where $\theta_X = \phi_X - \Sigma_{\mathcal{E}_\mathcal{X}}\Sigma_X^{-1}$. See also Section 2.6.5. Turning to the condition on the mean:

$$\begin{aligned} E[\eta_t] &= 0 \\ E[\beta' Y_t] &= \beta'(\mu_y + \phi_y E[Y_t]) = -e'_{\mathcal{O}}K^{\mathbb{P}}_{0X} + \delta' E[Y_t] = 0. \end{aligned} \tag{A.26}$$

Then, neglecting conditions involving restrictions on the covariance matrix, the following condition implies that η_t is a measurement error:

$$\delta = 0 \quad \text{and} \quad e'_{\mathcal{O}}K^{\mathbb{P}}_{0X} = 0. \tag{A.27}$$

These restrictions deliver a process where the consistency condition between expectation and survey data holds, but the canonical form in Theorem 1 must be adapted (available from the authors). If, in addition, $\gamma_{0\mathcal{O}} = \gamma_{0\mathcal{S}}$ and $\gamma_{\mathcal{O}} = \gamma_{\mathcal{S}}$, then the canonical form in Theorem 1 follows through but using the relabeled coefficients $\gamma_{0\mathcal{O}}$ and $\gamma_{\mathcal{O}}$ instead of γ_0 and γ_1. We modify Equation (18) as follows:

$$\mathcal{M}_t = \gamma_0 + \gamma_1\mathcal{Z}_t + \gamma_2\mathcal{E}_t, \tag{A.28}$$

with loadings given by

$$\gamma_0 = \begin{bmatrix} \gamma_{0\mathcal{O}} \\ \gamma_{0\mathcal{S}} \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} \gamma_{\mathcal{O}} \\ 0 \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} 0 \\ \gamma_{\mathcal{S}} \end{bmatrix}. \tag{A.29}$$

Then, these additional restrictions on the matrix C, D_1, and D_2 are given by

$$e'_{\mathcal{O}}C = e'_{\mathcal{S}}C = \gamma_{\pi 0}$$
$$e'_{\mathcal{O}}D_1 = e'_{\mathcal{S}}D_2 = \gamma'_{\pi}$$
$$e'_{\mathcal{S}}D_1 = e'_{\mathcal{O}}D_2 = 0.$$

This implies that

$$e'_{\mathcal{O}}K^{\mathbb{P}}_{0X} = 0 \qquad e'_{\mathcal{O}}(I + K^{\mathbb{P}}_{1X}) = e'_{\mathcal{S}} \qquad e'_{\mathcal{O}}\theta_X = 0, \tag{A.30}$$

and, in addition, that $\mathrm{var}(\eta_t) = 0$ (no measurement error). First consider $e'_{\mathcal{O}}K^{\mathbb{P}}_{0X} = 0$. From Equation (A.5),

$$\begin{aligned} e'_{\mathcal{O}}K^{\mathbb{P}}_{0X} &= e'_{\mathcal{O}}DK^{\mathbb{P}}_0 - e'_{\mathcal{O}}K^{\mathbb{P}}_{1X}\big(C + D_2K^{\mathbb{P}}_0\big)\gamma'_{\pi}K^{\mathbb{P}}_0 - (e_{\mathcal{S}} - e_{\mathcal{O}})'\big(C + D_2K^{\mathbb{P}}_0\big) \\ &= \gamma'_{\pi}K^{\mathbb{P}}_0 - (e_{\mathcal{S}} - e_{\mathcal{O}})'C + (e_{\mathcal{S}} - e_{\mathcal{O}})'D_2K^{\mathbb{P}}_0\gamma'_{\pi}K^{\mathbb{P}}_0 - 0 + \gamma'_{\pi}K^{\mathbb{P}}_0 = 0, \end{aligned} \tag{A.31}$$

where we used $K^{\mathbb{P}}_{1X} = e'_{\mathcal{X}} - e'_{\mathcal{O}}$. Note that $e'_{\mathcal{O}}D = e'_{\mathcal{O}}(D_1 + D_2\big(K^{\mathbb{P}}_1 + I_{\mathcal{N}_{\mathcal{Z}}}\big)) = \gamma'_{\pi}$. Then,

$$\begin{aligned} e'_{\mathcal{O}}K^{\mathbb{P}}_{1X} &= e'_{\mathcal{O}}DK^{\mathbb{P}}_X D^{-1} = \gamma'_{\pi}K^{\mathbb{P}}_X D^{-1} = e'_{\mathcal{S}}D_2K^{\mathbb{P}}_X D^{-1} \\ &= e'_{\mathcal{S}}(D - D_1 - D_2)D^{-1} = e'_{\mathcal{S}}(I_{\mathcal{N}_{\mathcal{Z}}} - D_1D^{-1} - D_2D^{-1}) \\ &= e'_{\mathcal{S}} - 0 - e'_{\mathcal{S}}D_2D^{-1} = e'_{\mathcal{S}} - \gamma'_{\pi}D^{-1} = e'_{\mathcal{S}} - e'_{\mathcal{O}}. \end{aligned} \tag{A.32}$$

Finally,

$$\begin{aligned} \mathcal{S}_t &= e'_{\mathcal{S}}\mathcal{X}_t = e'_{\mathcal{O}}\mathcal{E}_{\mathcal{X},t} = e'_{\mathcal{O}}I + K^{\mathbb{P}}_{1X} + e_{\mathcal{O}}\theta_{\mathcal{X}}\mathcal{E}_{\mathcal{X},t-1} \\ &= e'_{\mathcal{S}}\mathcal{X}_t + e_{\mathcal{O}}\theta_{\mathcal{X}}\mathcal{E}_{\mathcal{X},t-1} \\ \mathcal{S}_t &= \mathcal{S}_t + e_{\mathcal{O}}\theta_{\mathcal{X}}\mathcal{E}_{\mathcal{X},t-1}, \end{aligned} \tag{A.33}$$

which can be true only if $e_{\mathcal{O}}\theta_{\mathcal{X}} = 0$. This in turn implies that $e'_{\mathcal{O}}\Sigma_{\mathcal{E}_X} = e'_{\mathcal{S}}\Sigma_X$ and therefore that $e'_{\mathcal{S}}\Sigma^* = 0$.

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